QUARTIC CURVES MODULO 2*

BY

L. E. DICKSON

1. **Introduction.** Let $f(x, y, z)$ be a homogeneous form of order n with integral coefficients. The points for which the three partial derivatives of f are congruent to zero modulo 2 shall be called derived points. A derived point shall be called a singular point or an apex of $f = 0$ according as it is or is not on $f = 0$. Apices do not arise if n is odd, since the left member of Euler's relation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

is zero at a derived point and therefore also f is zero. But if n is even, a derived point may not be on $f = 0$ and thus be an apex.

For example, any non-degenerate conic modulo 2 can be transformed linearly into $x^2 + yz = 0$. Its single derived point (100) is an apex.

Quartic curves modulo 2 have the remarkable property of possessing at most seven bitangents (or an infinity in a special case), whereas an algebraic quartic curve possesses twenty-eight in general. For the special quartic β of § 4, any line through the apex (001) is a bitangent, just as any line through the apex of the conic $x^2 + yz = 0$ is a tangent.

The number of non-equivalent types of quartic curves containing 0, 7, 6 real points and having no real linear factor is 8, 1, 6, respectively. In each case, the types are completely distinguished by the number and reality of the singular points and apices. Except for two types, in which there are only two bitangents and only two derived points, the intersections of the bitangents coincide completely with the derived points. The problem is more complicated in the case of quartic curves with five real points, there being twenty-five types (§ 7). Quartics with 1, 2, 3, or 4 real points have not been treated since they would probably not present sufficient novelty to compensate for the increased length of the investigation.

2. **Bitangents.** The general quartic is

$$Q = ax^4 + by^4 + cz^4 + dx^3 y + exy^3 + fx^3 z + gxz^3 + hy^3 z + iyz^3 + jx^2 y^2 + kx^2 z^2 + ly^2 z^2 + mx^2 yz + nxy^2 z + pxyz^2.$$

* Presented to the Society (San Francisco Section), October 24, 1914.

Either (I) $d \equiv e \equiv f \equiv g \equiv h \equiv i \equiv 0 \pmod 2$ or, after permuting the variables, we may assume that $d \equiv 1 \pmod 2$. In the latter case we replace x by $x + ey$ and obtain a Q with $d = 1$, $e = 0$. Replacing y by $y + fz$, and x by $x + mz$, we have also $f = m = 0$. In this case (II), $d = 1$, $e = f = m = 0$, the line $x = ry + sz$ is a bitangent to Q if the quartic in y and z obtained by eliminating x is a perfect square, i. e., has no terms in $y^3 z$ and yz^3; the conditions are

$$r^2 s + nr + h = 0, \qquad s^3 + ps + gr + i = 0.$$

If $g \equiv 1 \pmod 2$, there are at most 7 sets of solutions r, s. Next, let $g \equiv 0 \pmod 2$. Any root $\neq 0$ of the cubic in s leads to at most two r's. For $i = 0$ the root $s = 0$ leads to an infinitude of r's if and only if $n = h = 0$. In this case (II), $y = \rho z$ is a bitangent if and only if $\rho = 0$, $g = 0$, while $z = 0$ is not a bitangent. *There are at most seven bitangents in case* (II), *unless* $g = i = h = n = 0$, *when there is an infinitude of bitangents, viz., all lines through* (001).

For case (I), $z = 0$ is a bitangent; also $y = \rho z$ if and only if $n\rho^2 + p\rho = 0$; while $x = ry + sz$ is a bitangent only when

$$mr^2 + nr = 0, \qquad ms^2 + ps = 0.$$

In case (I), *there is an infinity of bitangents if two of the coefficients* m, n, p *are zero, otherwise at most seven.*

3. **Real Points on the Quartic.** The seven real points (i. e., with integral coördinates) modulo 2 are

$$1 = (100), \quad 2 = (010), \quad 3 = (001), \quad 4 = (110), \quad 5 = (101),$$
$$6 = (011), \quad 7 = (111).$$

The values of Q at these points are respectively

$$a, b, c, \quad a + b + d + e + j, \quad a + c + f + g + k, \quad b + c + h + i + l,$$

and the sum of the fifteen coefficients.

4. **Quartic Curves Without Real Points.** The seven values in § 3 are now unity, so that

$$a = b = c = 1, \qquad j = d + e + 1, \qquad k = f + g + 1, \qquad l = h + i + 1,$$
$$p = m + n + 1.$$

(I) Consider case (I) of § 2. If m and n are not both zero, we may set $m = 1$, after interchanging x and y if necessary. If $m = n = 1$, we get

$$\alpha = x^4 + y^4 + z^4 + x^2 y^2 + x^2 z^2 + y^2 z^2 + x^2 yz + xy^2 z + xyz^2,$$

which is invariant under all real linear transformations. Its bitangents are the seven real lines. Their intersections, the seven real points, are apices and give all the apices. There is no singular point.

If $m = 1, n = 0$, we interchange x and z and have the case $m = n = 0$, viz.,

$$\beta = x^4 + y^4 + z^4 + x^2y^2 + x^2z^2 + y^2z^2 + xyz^2.$$

The only bitangents are $y = 0$, $z = 0$ and $x = ry$, where r is arbitrary. Their intersections are 1, 3, and $(r\,1\,0)$, all being apices except the last for $r^2 + r + 1 = 0$, when $(r\,1\,0)$ is a singular point. There are no further derived points. Any line through apex 3 is a bitangent.

(II) Consider case (II) of § 2, viz., $d = 1, e = f = m = 0$.

(1) First, let $h = 1$. If $n = 1$, we replace y by $y + x$ and then x by $x + z$ and obtain a quartic with $h = 1, n = 0$, viz.,

$$\gamma_{g,i} = x^4 + y^4 + z^4 + x^3y + gxz^3 + (g+1)x^2z^2 + y^3z + iyz^3 + iy^2z^2 + xyz^2.$$

By § 2, the bitangents are $y = 0$ (if $g = 0$) and

$$x = ry + sz, \qquad r^2s = 1, \qquad s^3 + s + gr + i = 0.$$

If $g = i = 0$, the bitangents are $y = 0$ and $x = y + z$, crossing at the apex 5; the only further derived point is the apex 3.

If $g = 0$, $i = 1$, the bitangents are $y = 0$, $x = (s+1)y + sy$, where $s^3 + s + 1 = 0$; they intersect at the apices 7 and $(s01)$. There is no further derived point.

If $g = 1, i = 0$, the seven bitangents are

$$x = ry + r^{-2}z, \qquad r^7 + r^4 + 1 \equiv 0 \pmod 2.$$

Replacing r by another root r^2 of this irreducible congruence, we obtain a bitangent meeting the former at $P = (\eta^3\,\eta 1)$, where $\eta = r^6 + r^2 + 1$ is a root of $\eta^7 + \eta + 1 = 0$. The seven roots of the latter are obtained by replacing r by r^2 in succession in η, the resulting function, etc. These seven apices P are the only derived points.

If $g = i = 1$, the bitangents are

$$x = ry + r^{-2}z, \qquad r = 1, \qquad r^2 + r + 1 = 0 \quad \text{or} \quad r^4 + r^3 + 1 = 0.$$

Their intersections are seen to be the apex 6 and the singular points

$$(y^3 + y, y, 1), \qquad y^2 + y + 1 = 0, \quad \text{or} \quad y^4 + y + 1 = 0.$$

There are no further derived points.

(2) Second, let $h = g = 0$. If $n = 1$, $i = 0$, we replace x, y, z by y, $x + z$, x and then x by $x + y$ and get γ_{00}. If $n = i = 0$, we replace z by $z + x$ and have case (I). If $n = 0$, $i = 1$, we replace x by $x + y$, z by $z + y$, and get γ_{01}. If $n = i = 1$, we have

$$\delta = x^4 + y^4 + z^4 + x^3 y + x^2 z^2 + yz^3 + xy^2 z .$$

By § 2, its bitangents are $y = 0$, $x = z$, $x = sz$, $x = y + z$, $x = s^{-1} y + sz$, where $s^2 + s + 1 = 0$. Their intersections are the apices 2, 5, 7 and the singular points $(s01)$, $(s1s^2)$. There are no further derived points.

(3) Finally, let $h = 0$, $g = 1$. If $i = 0$, we apply (xyz) and, in case $n = 0$, also $x' = x + z$, and obtain case (1). Hence let $i = 1$. Then, if $n = 0$, we replace y by $y + x$ and have case (2). But if $n = 1$, we have

$$\epsilon = x^4 + y^4 + z^4 + x^3 y + xz^3 + yz^3 + xy^2 z .$$

By § 2, its bitangents are $x = z$, $x = sz$, $x = \sigma^{-1} y + \sigma z$, where

$$s^2 + s + 1 = 0, \qquad \sigma^4 + \sigma + 1 = 0 .$$

Their intersections are the apex 2 and the singular points $(1\tau 1)$ and $(\tau\sigma^3 1)$ where $\tau = \sigma + \sigma^2$ is a root of $\tau^2 + \tau + 1 = 0$. There are no further derived points.

The eight classes of quartic curves modulo 2 without real points are distinguished by the number and reality of their apices and singular points. These derived points coincide with the intersections of the bitangents, except for γ_{00} which has only two bitangents and two apices.

5. **Quartic Curves Containing all Seven Real Points.** The conditions are

$$a = b = c = 0, \qquad j = d + e, \qquad k = f + g, \qquad l = h + i, \qquad p = m + n .$$

For case (I) of § 2, Q is the product of four linear functions. For case (II),

$$Q = x^3 y + x^2 y^2 + gxz^3 + gx^2 z^2 + hy^3 z + iyz^3 + (h + i) y^2 z^2 + nxy^2 z + nxyz^2 .$$

It has the factor x if $h = i = 0$, factor y if $g = 0$, factor $x + y$ if $i = 1$, $h = n$, factor $x + z$ if $h = 0$, $n + i = 1$, factor $x + y + z$ if $n = i$, $h + i = 1$. We exclude these cases. Then, if $n = 0$, we have $h = i = 1$ and get

$$A = x^3 y + x^2 y^2 + xz^3 + x^2 z^2 + y^3 z + yz^3 .$$

But, if $n = 1$, we have $h = 1$, $i = 0$; the resulting Q is derived from A by replacing x by $y + z$, y by x, z by y. The bitangents to A are

$$x = (s^3 + 1) y + sz, \qquad s^7 + s + 1 = 0 .$$

Their intersections are the apices $(1z^3 z)$, $z^7 + z^3 + 1 = 0$, which give all the derived points. *The single type of irreducible quartic curve with seven real points has seven imaginary bitangents whose seven intersections are apices and give all the derived points.*

6. **Quartic Curves with Six Real Points.** After applying a real transformation, we may assume that the curve contains the real points other than 1. Then

$a=1, \quad b=c=0, \quad d+e+j=f+g+k=m+n+p=1, \quad l=h+i.$

(1) First, let d and f be not both zero. As in § 2, we may set $d=1$, $e=f=m=0$. Then Q has the factor x if $h=i=0$, factor $x+y$ if $h=n$, $g=i$, factor $x+z$ if $h=0$, $n=i$, factor $x+y+z$ if $h=n+1$, $g=i+n$. We exclude these cases. Then if $h=0$, we have $i=n=g=0$, and get

$$B=x^4+x^3y+x^2z^2+yz^3+y^2z^2+xyz^2.$$

Its four bitangents are $y=0$ and $x=sz$, where $s^3+s+1=0$. Their intersections are the singular point 2 and the apices $(s01)$. There are no further derived points.

Next, if $h=n=1$, we have $i=g+1$. According as $g=0$ or $g=1$, we get

$$C=x^4+x^3y+x^2z^2+y^3z+yz^3+xy^2z,$$

$$D=x^4+x^3y+xz^3+y^3z+y^2z^2+xy^2z.$$

The seven bitangents to C are $y=0$, $x=ry+z$, where $r^2+r+1=0$, and

$$x=\rho y+(\rho+\rho^3)z, \qquad \rho^4+\rho^3+1=0.$$

Their intersections are the singular point 5 and the apices $(r01)$, and $(x1x^2)$ where $x^4+x+1=0$. There are no further derived points.

The seven bitangents to D are $x=s^3y+sz$, $s^7+s^3+1=0$. This meets the one with s replaced by the root s^2 in $P=(x1z)$, where $x=s^5+s^4$, $z=s^4+s^3+s^2$, so that $z=x^3$, $x^7+x+1=0$. All seven roots of the latter are obtained by replacing s by s^2 in succession in x, the resulting function, etc. These seven apices P give all the derived points of D.

Finally, if $h=1$, $n=0$, we have $i=g+1$. Then if $g=0$ the function is derived from B by the transformation $x'=x+y$, $z'=z+y$. If $g=1$, it is derived from D by $z'=z+y$.

(2) Second, let $d=f=0$, while e and g are not both zero. Interchanging y and z, if necessary, we may set $e=1$. We make $h=n=0$ by use of $x'=x+z$ and $y'=y+z$. We may set $i=1$, $g=m$, since otherwise Q has the factor x, $x+y$, or $x+y+z$. According as $m=0$ or $m=1$, we have

$$E=x^4+xy^3+x^2z^2+yz^3+y^2z^2+xyz^2,$$

$$F=x^4+xy^3+xz^3+yz^3+y^2z^2+x^2yz.$$

The only bitangents to E are $y=0$ and $y=z$. They intersect at the apex 1. The only further derived point is the singular point 5.

The seven bitangents to F are $y=\alpha z$, where $\alpha^3=1$, and

$$x=ry+r^2z, \qquad r^4+r+1=0.$$

Their intersections are the four singular points

$$(x, x^2 + x^3, 1), \qquad x^4 + x^3 + x^2 + x + 1 = 0,$$

and the apices 1 and $(x, 1, 1)$, where $x^2 + x + 1 = 0$. There are no further derived points.

(3) Third, let $d = f = e = g = 0$. If $m = 0$, we make $e = 1$ by use of $z' = z + y$ and have case (2). Henceforth, let $m = 1$. If $n = 0$, Q has the factor $x + hy + iz$. Hence we may set $n = 1$. If $i = h = 0$, Q has the factor x. In the contrary case, we interchange y and z and set $h = 1$. By use of $y' = y + z$, we make $i = 0$ and get

$$G = x^4 + x^2 y^2 + x^2 z^2 + y^3 z + y^2 z^2 + x^2 yz + xy^2 z + xyz^2.$$

Its seven bitangents are $y = 0$, $z = 0$, $y = z$, $x = ry$, $x = ry + z$, where $r^2 + r + 1 = 0$. Their intersections are the singular points 3, 5 and the apices 1, $(r10)$, $(r11)$. There are no further derived points.

The six types of quartic curves containing six real points and having no linear factor are distinguished by the number of singular points and apices. These derived points coincide with the intersections of the bitangents, except for E which has only two bitangents and two derived points.

7. **Quartic Curves with Five Real Points.** The two real points not on the curve may be taken to be 1 and 2. Then

$$a = b = 1, \qquad c = 0, \qquad j = d + e, \qquad k = f + g + 1, \qquad l = k + i + 1,$$

$$p = m + n.$$

(I) Let d and e be not both zero. Interchanging x and y if necessary, we may set $d = 1$. Applying $x' = x + z$ and $y' = y + z$, we may set $m = f = 0$ and get

$$q = x^4 + y^4 + x^3 y + exy^3 + (e + 1)x^2 y^2 + gxz^3 + (g + 1)x^2 z^2 + hy^3 z + iyz^3 + (h + i + 1)y^2 z^2 + nxy^2 z + nxyz^2.$$

A real transformation replacing q by a function of the same type must leave unaltered the line $z = 0$ determined by the two real points 1 and 2. But $z = 0$ meets $q = 0$ at $(x10)$, where $(x + 1)^2(x^2 + x + 1) = 0$ if $e = 1$, and $(x + 1)(x^3 + x + 1) = 0$ if $e = 0$; while $z = 0$ meets Q with $d = e = 0$ at $(x10)$ where $(x + 1)^4 = 0$. Hence quartics belonging to different ones of these three classes are not equivalent.

(I_1) Let $e = 1$. We may drop the case $h = 1$, $n = 0$; for, if we interchange x and y and then replace y by $y + z$, we obtain a q with the coefficients $e' = 1$, $g' = i + 1$, $h' = 0$, $i' = g$, $n' = 1$. If $h = n$, we may drop the case $i = 0$, $g = 1$; for, if $h = n = 0$, we have only to interchange x and y to permute

g and i; while, if $h = n = 1$, we replace x by $y + z$ and y by $x + z$ and obtain a like q with g and i permuted. We set

$$q = [ghin], \qquad \mu = x^4 + y^4 + x^3 y + xy^3.$$

The q's with a single singular point S and no apex are

$$[1010] = \mu + xz^3 + yz^3, \qquad S = 4;$$

$$[1001] = \mu + xz^3 + y^2 z^2 + xy^2 z + xyz^2, \qquad S = 6.$$

These are not equivalent since 1 and 2 are collinear with 4, but not with 6.

Those with exactly four derived points are

$$[1111] = \mu + xz^3 + y^3 z + yz^3 + y^2 z^2 + xy^2 z + xyz^2, \qquad S = 4, 6, (11z), \quad z^2 + z + 1 = 0.$$

$$[0101] = \mu + x^2 z^2 + y^3 z + xy^2 z + xyz^2, \qquad S = 3, 4, 5, 7.$$

$$[0001] = \mu + x^2 z^2 + y^2 z^2 + xy^2 z + xyz^2, \qquad S = 3, 5; \quad \text{apices } (01z),\ z^2 + z + 1 = 0.$$

$$[0010] = \mu + x^2 z^2 + yz^3, \qquad S = 4, 5; \qquad \text{apices } (x01),\ x^2 + x + 1 = 0.$$

The last two are not equivalent, since 4 alone is collinear with 1, 2.

Those with one real and six imaginary derived points are

$$[0h11] = \mu + x^2 z^2 + hy^3 z + yz^3 + hy^2 z^2 + xy^2 z + xyz^2.$$

If $h = 1$, the only singular point is 4, while the apices are $(z01)$, $(z^2 + 1, 1, z)$, where $z^3 + z + 1 = 0$. If $h = 0$, the singular points are 7 and $(z^2\, 1z)$, where $z^3 + z^2 + 1 = 0$, while the apices are $(x01)$, $x^3 + x + 1 = 0$.

The only q with seven imaginary derived points (apices A) is

$$[1011] = \mu + xz^3 + yz^3 + xy^2 z + xyz^2, \qquad A = \left(\frac{z^2}{z^2 + 1}, 1, z\right), \quad z^7 + z^6 + z^5 + z^3 + z^2 + z + 1 = 0.$$

There remains only [0000], which has the factor $x + y$.

(I_2) Let $e = 0$. Write

$$q = (ghin), \qquad \lambda = x^4 + y^4 + x^3 y + x^2 y^2.$$

The part λ of q free of z has the single real factor $x + y$. Hence the only possible real linear factors of q are $x + y$ and $x + y + z$. The first is a factor if $g = i$, $n = h$; the second if $g = i + h$, $n = h + 1$. After the exclusion of these cases, there remain only the following eight, for which all of the derived points are shown.

$(0100) = \lambda + x^2z^2 + y^3z$, singular 3.

$(1000) = \lambda + xz^3 + y^2z^2$, apex 2.

$(1001) = \lambda + xz^3 + y^2z^2 + xy^2z + xyz^2$, singular 7, apices 2, $(01z)$, $z^2 + z + 1 = 0$.

$(0010) = \lambda + x^2z^2 + yz^3$, singular 5, apices 2, $(x01)$, $x^2 + x + 1 = 0$.

To show that the last two are not equivalent, we consider the tangents at the real points of the curve other than 4 and the singular point. For the first,* the tangents at these points 3, 5, 6 are $x = 0$, $x = z$, $x = 0$; for the second, the tangents at 3, 6, 7 are $y = 0$, $y = z$, $x = z$ and are distinct.

$(0011) = \lambda + x^2z^2 + yz^3 + xy^2z + xyz^2$, apices 2, $(z01)$, $(z^2\,1z)$, $z^3 + z + 1 = 0$.

$(0111) = \lambda + x^2z^2 + y^3z + yz^3 + y^2z^2 + xy^2z + xyz^2$, singular 6, apices $(x01)$, $x^3 + x + 1 = 0$, $(z^2 + 1, 1, z)$, $z^3 + z^2 + 1 = 0$.

$(1110) = \lambda + xz^3 + y^3z + yz^3 + y^2z^2$, apices $(1 + z^{-2}, 1, z)$, $z^7 + z^4 + 1 = 0$.

$(1101) = \lambda + xz^3 + y^3z + xy^2z + xyz^2$, apices $\left(\frac{1}{z^2+1}, 1, z\right)$, $z^7 + z^6 + z^5 + z^3 + z^2 + z + 1 = 0$.

To show that the last two are not equivalent, we employ the tangent at the point 4 uniquely determined by the real points 1 and 2. For the first, this tangent is $x + y + z = 0$; it meets the curve at 4, 5, 6; the tangent at 5 is $x = z$ and meets the curve again at 7; the tangent at 6 is $x = 0$ and meets the curve again at 3. For (1101), the tangent at 4 is $x = y$, which meets the curve at 4, 3, 7; the tangents at 3 and 7 are $x = 0$ and $y = z$, each of which meets the curve again only at 6.

(II) Let d and e be zero, but f and h not both zero. After interchanging x and y if necessary, we may set $f = 1$, and then make $g = 0$ by use of $x' = x + z$. We get

* Its bitangents are $x = 0$, $x = z$, $x = s^2y + sz$, $s^2 + s + 1 = 0$. The bitangents to (0010) are $y = 0$, $x = z$, $x = sz$. In each case their intersections are the derived points.

$$Q = x^4 + y^4 + x^3 z + hy^3 z + iyz^3 + (h + i + 1) y^2 z^2 + mx^2 yz + nxy^2 z + (m + n) xyz^2.$$

The possible real linear factors are $x + y$, $x + y + z$, occurring if

$$h = m + n + 1, \quad i = 0 \text{ or } i = n + 1.$$

(II_1) Let $m = 0$, $h = 1$. We make $i = 0$ by use of $y' = y + z$. If $n = 0$, $x + y$ is a factor. Hence we set $n = 1$ and get

$$x^4 + y^4 + x^3 z + y^3 z + xy^2 z + xyz^2, \quad \text{singular } 3, \quad \text{apices } (x10), (x^2 x1), \quad x^3 + x + 1 = 0.$$

(II_2) Let $m = h = 0$. We exclude the case $i = 0$, $n = 1$, when $x + y$ is a factor. If $n = i = 0$, we have

$$x^4 + y^4 + x^3 z + y^3 z^2, \quad \text{singular } 3, 6, \quad \text{apices } 2, (0y1), \quad y \text{ any } \neq 0, 1.$$

If $n = 0$, $i = 1$, we have

$$x^4 + y^4 + x^3 z + yz^3, \quad \text{apex 2 only derived point.}$$

If $n = i = 1$, we have

$$x^4 + y^4 + x^3 z + yz^3 + xy^2 z + xyz^2, \quad \text{singular } 4, \quad \text{apices } 2, (1y1), \quad y^2 + y + 1 = 0.$$

(II_3) Let $m = 1$. We exclude the case $h = n$, $i = 0$ or $i = n + 1$, since Q then has the factor $x + y$ or $x + y + z$. First, let $n = 0$, $h = 1$. The form with $i = 1$ is obtained from that with $i = 0$ by replacing x by $x + z$, y by $y + z$. In that with $i = 0$ we interchange x and y and get the form in (II_1). Next, if $n = h = i = 1$, we have

$$x^4 + y^4 + x^3 z + y^3 z + yz^3 + y^2 z^2 + x^2 yz + xy^2 z, \quad \text{singular } 4, \quad \text{no apex.}$$

Finally, let $n = 1$, $h = 0$. If $i = 0$, we have

$$x^4 + y^4 + x^3 z + y^2 z^2 + x^2 yz + xy^2 z, \quad \text{singular } 3, \quad \text{apices } 2, (x10), \quad x^2 + x + 1 = 0.$$

Replacing x by $x + z$, y by $y + z$, we obtain the form with $i = 1$.

(III) $d = e = f = h = 0$. If $m = n$, $g = i$ or $g = i + n$, Q has the factor $x + y$ or $x + y + z$. First, let m and n be not both zero. Interchanging x and y, we may set $m = 1$. If $n = 1$, Q has a linear factor. If $n = 0$, we make $g = 0$ by use of $y' = y + z$. According as $i = 1$ or $i = 0$, we have

$x^4 + y^4 + x^2 z^2 + yz^3 + x^2 yz + xyz^2$, apices 1, 2, $(x01)$,

$x^2 + x + 1 = 0$;

$x^4 + y^4 + x^2 z^2 + y^2 z^2 + x^2 yz + xyz^2$, singular 3, 5, apices 1, 2.

Second, let $m = n = 0$. There is a linear factor if $g = i$. In the contrary case we interchange x and y and have $g = 1$, $i = 0$, and get

$x^4 + y^4 + xz^3 + y^2 z^2$, singular 4, apices 2, $(1y0)$, y any $\neq 1$.

The only types in (II) and (III) for which there is the same number of singular points and same number of apices are the last ones in (II_2) and (II_3). These are not equivalent since the real points 1 and 2, not on the curve, are collinear with the singular point 4 of the first curve, but not with that of the second.

The 25 types of quartic curves containing the five real points other than 1 and 2, and having no real linear factor, are distinguished by the number and reality of their singular points and apices, and their relation to the line joining 1 and 2, together with properties of the tangents at the real points in the cases of two pairs of quartics in (I_2).

THE UNIVERSITY OF CALIFORNIA,
October, 1914.

MIXED LINEAR INTEGRAL EQUATIONS OF THE FIRST ORDER*

BY

WALLIE ABRAHAM HURWITZ

In a previous paper,† the author has defined a mixed‡ integral equation as an equation in which an unknown function may be involved under integral signs operating over regions of different numbers of dimensions and in which the value of the function at special points may occur. The highest number of dimensions of any integral was called the order of the equation. A theorem was proved concerning mixed *linear* equations of the first order. It is the purpose of the present paper to give a systematic treatment of this case.§ A complete theory of a special form of such equations, together with interesting applications to physical problems, has been given by Kneser.‖

Mixed linear integral equations are of interest, not only because they present themselves naturally in the attempt to solve certain problems in connection with differential equations, but also because they furnish another chapter in the theory of equations involving linear operations. As might be expected, they follow closely the analogy of linear algebraic equations and pure linear integral equations; but the generalizations necessary to exhibit the analogy are sometimes by no means trivial. Thus, a certain lack of symmetry pointed out at the close of the earlier paper has now been removed by the device of considering as *adjoint* to the given equation (which involves one function as the only unknown) a system of equations, involving one unknown function and several unknown constants.

1. We consider the homogeneous mixed linear integral equation

$$u(x) = \sum_{i=1}^{m} K^{(i)}(x)\, u(\xi_i) + \int_a^b K(x, s)\, u(s)\, ds \tag{1}$$

* Presented to the Society, December 31, 1912.

† Bulletin of the American Mathematical Society, vol. 18 (1912), p. 291.

‡ The term "mixed equation" has been used in a different sense by Bratu, Paris Comptes Rendus, vol. 148 (1909), p. 1370.

§ The next higher case, in which appear integrals over two- and one-dimensional regions, as well as values of the function at special points, has been studied by Dr. J. Rosenbaum, in a paper to be published soon.

‖ Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), p. 169. In terms of the notation of the present paper, the special case considered by Kneser is that in which $K^{(i)}(x) = M_i K(x, \xi_i)$, where the quantities M_i are positive constants.

and the non-homogeneous equation

$$u(x) = f(x) + \sum_{i=1}^{m} K^{(i)}(x)\, u(\xi_i) + \int_a^b K(x, s)\, u(s)\, ds. \tag{2}$$

The points $\xi_1, \xi_2, \cdots, \xi_m$ are any distinct points of the interval $a \leqq x \leqq b$. The given functions $f(x)$, $K^{(1)}(x), \cdots, K^{(m)}(x)$ will be assumed to be continuous, $a \leqq x \leqq b$; and the function $K(x, y)$ will be assumed to be continuous, $a \leqq x \leqq b$, $a \leqq y \leqq b$. We speak of the set of functions

$$[K(x, y);\ K^{(1)}(x), \cdots, K^{(m)}(x)]$$

as the *kernel system* of the equations (1), (2). By a *solution* $u(x)$ we shall understand without further specification a continuous solution. The solution $u(x) = 0$ of (1) will be termed a *trivial solution.*

It is convenient to consider with equations (1), (2) the *adjoint homogeneous system* of equations

$$\begin{aligned} v(y) &= \sum_{i=1}^{m} v^{(i)}\, K(\xi_i, y) + \int_a^b v(s)\, K(s, y)\, ds, \\ v^{(p)} &= \sum_{i=1}^{m} v^{(i)}\, K^{(p)}(\xi_i) + \int_a^b v(s)\, K^{(p)}(s)\, ds \qquad (p = 1, 2, \cdots, m) \end{aligned} \tag{1'}$$

and the non-homogeneous system

$$\begin{aligned} v(y) &= g(y) + \sum_{i=1}^{m} v^{(i)}\, K(\xi_i, y) + \int_a^b v(s)\, K(s, y)\, ds, \\ v^{(p)} &= g^{(p)} + \sum_{i=1}^{m} v^{(i)}\, K^{(p)}(\xi_i) + \int_a^b v(s)\, K^{(p)}(s)\, ds \qquad (p = 1, 2, \cdots, m). \end{aligned} \tag{2'}$$

Here $g(y)$ is a given continuous function, $a \leqq y \leqq b$, and $g^{(1)}, \cdots, g^{(m)}$ are given constants. By a *solution* is understood a system consisting of a continuous function $v(y)$ and m constants $v^{(1)}, \cdots, v^{(m)}$, satisfying the equations. We term the solution of (1′)

$$v(y) = 0, \qquad v^{(1)} = \cdots = v^{(m)} = 0$$

a *trivial solution.* If a number of solutions are so related that a linear combination of them, not all coefficients being equal to zero, yields a trivial solution, they are said to be *linearly dependent;** otherwise *linearly independent.*

Let the normal orthogonal sets of principal solutions for $K(x, y)$ regarded as kernel of a pure integral equation be $\phi_1(x), \cdots, \phi_n(x)$; $\psi_1(y), \cdots, \psi_n(y)$; then

* The same constant multipliers must be used for the functions contained in the solutions as for the sets of constants.

$$\phi_j(x) = \int_a^b K(x,s)\phi_j(s)\,ds \qquad (j = 1, 2, \cdots, n),$$

$$\psi_j(y) = \int_a^b \psi_j(s)K(s,y)\,ds \qquad (j = 1, 2, \cdots, n),$$

$$\int_a^b \phi_j(s)\phi_q(s)\,ds = \int_a^b \psi_j(s)\psi_q(s)\,ds = \begin{cases} 0, & q \neq j. \\ 1, & q = j. \end{cases}$$

The case in which no principal solutions exist is included by making the evident interpretation of the symbols involved when $n = 0$.

There exists a continuous function $k(x,y)$ such that†

$$k(x,y) = K(x,y) + \int_a^b K(x,s)k(s,y)\,ds - \sum_{j=1}^{n} \psi_j(x)\psi_j(y),$$

$$k(x,y) = K(x,y) + \int_a^b k(x,s)K(s,y)\,ds - \sum_{j=1}^{n} \phi_j(x)\phi_j(y).$$

The equation

$$u(x) = f(x) + \int_a^b K(x,s)u(s)\,ds \tag{3}$$

has no solutions unless

$$\int_a^b \psi_j(s)f(s)\,ds = 0 \qquad (j = 1, 2, \cdots, n); \tag{4}$$

if this condition is satisfied, solutions exist and may be written in the form

$$u(x) = f(x) + \int_a^b k(x,s)f(s)\,ds + \sum_{j=1}^{n} a_j\phi_j(x), \tag{5}$$

where $a_1, \cdots, a_n$ are constants. Similarly the equation

$$v(y) = g(y) + \int_a^b v(s)K(s,y)\,ds \tag{3'}$$

has no solutions unless

$$\int_a^b g(s)\phi_j(s)\,ds = 0 \qquad (j = 1, 2, \cdots, n); \tag{4'}$$

if this condition is satisfied, solutions exist and may be written in the form

$$v(y) = g(y) + \int_a^b v(s)k(s,y)\,ds + \sum_{j=1}^{n} b_j\psi_j(y), \tag{5'}$$

where $b_1, \cdots, b_n$ are constants.

2. THEOREM I. *Any solution $u(x)$ of (1) is expressible in the form*

$$u(x) = \sum_{i=1}^{m} A_i\left[K^{(i)}(x) + \int_a^b k(x,s)K^{(i)}(s)\,ds\right] + \sum_{j=1}^{n} B_j\phi_j(x), \tag{6}$$

† W. A. Hurwitz, these Transactions, vol. 13 (1912), p. 408.

where the $m+n$ constants $A_1, \cdots, A_m$; $B_1, \cdots, B_n$ satisfy the system of $m+n$ linear algebraic equations:

$$\sum_{i=1}^{m} A_i \int_a^b \psi_q(s) K^{(i)}(s)\, ds = 0 \qquad (q = 1, 2, \cdots, n), \tag{7}$$

$$-A_p + \sum_{i=1}^{m} A_i \left[K^{(i)}(\xi_p) + \int_a^b k(\xi_p, s) K^{(i)}(s)\, ds \right] + \sum_{j=1}^{n} B_j \phi_j(\xi_p) = 0 \qquad (p = 1, 2, \cdots, m).$$

Conversely, if $A_1, \cdots, A_m$; $B_1, \cdots, B_n$ satisfy (7), *the function $u(x)$ given by* (6) *is a solution of* (1).

To prove the first part of the theorem, suppose that (1) possesses a solution $u(x)$, and write $u(\xi_i) = A_i$; we have then

$$u(x) = \sum_{i=1}^{m} A_i K^{(i)}(x) + \int_a^b K(x, s) u(s)\, ds. \tag{8}$$

Conditions (4) yield the first set of equations (7). If we next write out $u(x)$ by use of (5), we have (6), and the substitution of ξ_p for x then gives the last set of (7). Conversely, if $[A_1, \cdots, A_m; B_1, \cdots, B_n]$ represents a solution of (7), then the first set of (7) furnishes a sufficient condition that $u(x)$ should satisfy (8); the second set shows that $u(\xi_p) = A_p$, thus identifying (8) with (1).

COROLLARY I. *The correspondence between solutions of* (1) *and solutions of* (7) *is unique.*

This statement will be proved if we demonstrate the following:

COROLLARY II. *To a trivial solution of* (1) *corresponds a trivial solution of* (7); *and conversely.*

That a trivial solution of (7) leads to a trivial solution of (1) is evident from the form of (6). To prove the other part of the proposition, suppose in (6) $u(x) = 0$; then as $u(\xi_p) = A_p$,

$$A_p = 0 \qquad (p = 1, 2, \cdots, m),$$

and

$$0 = \sum_{j=1}^{n} B_j \phi_j(x);$$

therefore on account of the linear independence of $\phi_1(x), \cdots, \phi_n(x)$, $B_q = 0$ $(q = 1, 2, \cdots, n)$.

As an immediate deduction, we have

COROLLARY III. *To linearly independent solutions of* (1) *correspond linearly independent solutions of* (7); *and conversely.*

3. Turning now to the adjoint system (1′), we find similarly

THEOREM I'. *Any solution* $[v(y); v^{(1)}, \cdots, v^{(m)}]$ *of* (1') *is expressible in the form*

$$(6') \qquad v(y) = \sum_{j=1}^{n} C_j \psi_j(y) + \sum_{i=1}^{m} D_i k(\xi_i, y), \qquad (p = 1, 2, \cdots, m),$$
$$v^{(p)} = D_p$$

where the $m + n$ *constants* $C_1, \cdots, C_n$; $D_1, \cdots, D_m$ *satisfy the system of* $m + n$ *linear algebraic equations*:

$$(7') \qquad \sum_{j=1}^{n} C_j \int_a^b \psi_j(s) K^{(p)}(s)\, ds - D_p + \sum_{i=1}^{m} D_i \Big[K^{(p)}(\xi_i) + \int_a^b k(\xi_i, s) K^{(p)}(s)\, ds \Big] = 0 \quad (p = 1, 2, \cdots, m),$$
$$\sum_{i=1}^{m} D_i \phi_q(\xi_i) = 0 \qquad (q = 1, 2, \cdots, n).$$

Conversely, if $C_1, \cdots, C_n$; $D_1, \cdots, D_m$ *satisfy* (7'), *the system*

$$[v(y); v^{(1)}, \cdots, v^{(m)}]$$

given by (6') *is a solution of* (1').

As before, suppose that we have a solution of (1'), and write $v^{(p)} = D_p$ as in (6'). Then

$$(8') \qquad v(y) = \sum_{i=1}^{n} D_i K(\xi_i, y) + \int_a^b v(s) K(s, y)\, ds.$$

The condition (4') that (8') be possible gives the last set of (7'). Then writing out the value of $v(y)$ by (5') and also applying the relation between $k(x, y)$ and $K(x, y)$,

$$\begin{aligned} v(y) &= \sum_{i=1}^{n} D_i \Big[K(\xi_i, y) + \int_a^b K(\xi_i, s) k(s, y)\, ds \Big] + \sum_{j=1}^{m} C_j' \psi_j(y) \\ &= \sum_{i=1}^{n} D_i k(\xi_i, y) + \sum_{i=1}^{n} \sum_{j=1}^{m} D_i \psi_j(\xi_i) \psi_j(y) + \sum_{j=1}^{m} C_j' \psi_j(y) \\ &= \sum_{i=1}^{n} D_i k(\xi_i, y) + \sum_{j=1}^{m} C_j \psi_j(y), \end{aligned}$$

as demanded by (6'). Substitution of the forms for $v(y)$ and $v^{(p)}$ in the second equation of (1') now yields the first set of (7'). The converse is also readily shown, as in the previous case. We have further, in analogy with the earlier results:

COROLLARY I. *The correspondence between solutions of* (1') *and solutions of* (7') *is unique.*

COROLLARY II. *To a trivial solution of* (1') *corresponds a trivial solution of* (7'); *and conversely.*

COROLLARY III. *To linearly independent solutions of* (1′) *correspond linearly independent solutions of* (7′); *and conversely.*

4. We have thus reduced the study of solutions of (1), (1′) to the study of solutions of the systems of linear homogeneous algebraic equations (7), (7′) respectively. The number of linearly independent solutions is in each case finite, being limited by the number of equations. Furthermore, the matrices of the coefficients of (7) and (7′) are conjugate.* Thus the existence of a non-trivial solution of either implies the existence of a non-trivial solution of the other, and both possess the same number of linearly independent solutions. We restate these facts for (1), (1′):

THEOREM II. *If either equation* (1) *or system* (1′) *possesses a non-trivial solution, the other possesses a non-trivial solution; each possesses a finite number of linearly independent solutions, and the number is the same in both cases.*†

5. If we apply the process used in the proof of Theorem I to the non-homogeneous equation (2), we obtain in place of (7) a non-homogeneous algebraic system for $A_1, \cdots, A_m$; $B_1, \cdots, B_n$, the matrix of the coefficients being the same as in (7). If (1) possesses no non-trivial solutions, then (7) possesses no non-trivial solutions; hence the corresponding non-homogeneous algebraic system possesses one and only one solution, and therefore (2) possesses one and only one solution. Similarly, we see that if (1′) has no non-trivial solutions, (2′) has one and only one solution.

From the method just described by which the solutions may be obtained it is evident that in case the function $f(x)$ of (2), or the function $g(y)$ and constants $g^{(1)}, \cdots, g^{(m)}$ of (2′) contain parameters, the solutions will be continuous functions of all variables involved so long as the same is true of the given functions.

In equation (2), write $f(x) = K(x, y)$, regarding y as a parameter; calling the solution $u(x) = Q(x, y)$, we see that in case (1) has no non-trivial solutions, there exists a continuous function $Q(x, y)$, such that

$$(9) \quad Q(x, y) = K(x, y) + \sum_{i=1}^{m} K^{(i)}(x) Q(\xi_i, y) + \int_a^b K(x, s) Q(s, y)\, ds.$$

Again, writing $f(x) = K^{(p)}(x)$, we demonstrate the existence of continuous functions $Q^{(p)}(x)$, such that

$$(10) \quad Q^{(p)}(x) = K^{(p)}(x) + \sum_{i=1}^{m} K^{(i)}(x) Q^{(p)}(\xi_i) + \int_a^b K(x, s) Q^{(p)}(s)\, ds.$$

In the same manner, writing in (2′) $g(y) = K(x, y)$, $g^{(p)} = K^{(p)}(x)$,

* This fact and its consequences justify the designation of the system (1′) as adjoint to the equation (1).

† It may be readily shown that the number ν of linearly independent solutions satisfies the condition $n - m \leqq \nu \leqq n + m$.

where x is to be regarded as a parameter, we see that there exist continuous functions $R(x, y)$, $R^{(p)}(x)$, such that

$$(9') \quad R(x, y) = K(x, y) + \sum_{i=1}^{m} R^{(i)}(x) K(\xi_i, y) + \int_a^b R(x, s) K(s, y)\, ds,$$

$$(10') \quad R^{(p)}(x) = K^{(p)}(x) + \sum_{i=1}^{m} R^{(i)}(x) K^{(p)}(\xi_i) + \int_a^b R(x, s) K^{(p)}(s)\, ds.$$

The formulæ (9), (10), (9′), (10′) are generalizations of the familiar resolvent formulæ of pure integral equations. It will shortly be seen that the functions R and Q are identical; postponing the proof of this fact, we show how the solutions of (2), (2′) may be expressed in terms of them. Starting with equation (2), which we know to possess a solution, compute directly by substitution of the right-hand side of (2) for $u(x)$ the expression

$$\sum_{p=1}^{m} R^{(p)}(x) u(\xi_p) + \int_a^b R(x, s) u(s)\, ds;$$

after simplifying by use of (9′), (10′), as in the case of pure integral equations, we find

$$\sum_{i=1}^{m} K^{(i)}(x) u(\xi_i) + \int_a^b K(x, s) u(s)\, ds = \sum_{i=1}^{m} R^{(i)}(x) f(\xi_i) + \int_a^b R(x, s) f(s)\, ds.$$

Hence the solution of (2) is

$$(11) \quad u(x) = f(x) + \sum_{i=1}^{m} R^{(i)}(x) f(\xi_i) + \int_a^b R(x, s) f(s)\, ds.$$

Similarly the solution of (2′) is given by the formulæ:

$$(11') \quad \begin{aligned} v(y) &= g(y) + \sum_{i=1}^{m} g^{(i)} Q(\xi_i, y) + \int_a^b g(s) Q(s, y)\, ds, \\ v^{(p)} &= g^{(p)} + \sum_{i=1}^{m} g^{(i)} Q^{(p)}(\xi_i) + \int_a^b g(s) Q^{(p)}(s)\, ds \\ &\qquad (p = 1, 2, \cdots, m). \end{aligned}$$

We may prove the identity of the functions Q, R, as follows. Equations (9′), (10′) together constitute a system of the form (2′), the unknowns being

$$v(y) = R(x, y), \qquad v^{(p)} = R^{(p)}(x),$$

and the known functions

$$g(y) = K(x, y), \qquad g^{(p)} = K^{(p)}(x).$$

Writing out the solution in the form (11′),

$$R(x,y) = K(x,y) + \sum_{i=1}^{m} K^{(i)}(x)\,Q(\xi_i, y) + \int_a^b K(x,s)\,Q(s,y)\,ds,$$

$$R^{(p)}(x) = K^{(p)}(x) + \sum_{i=1}^{m} K^{(i)}(x)\,Q^{(p)}(\xi_i) + \int_a^b K(x,s)\,Q^{(p)}(s)\,ds \qquad (p = 1, 2, \cdots, m);$$

but the right-hand sides of these equations are equal respectively to the values given by (9) and (10) for $Q(x,y)$ and $Q^{(p)}(x)$.

6. We collect the results just deduced in the following theorem:

THEOREM III. *If* (1), (1′) *have no non-trivial solutions, then there exists a resolvent system*

$$[Q(x,y);\ Q^{(1)}(x), \cdots, Q^{(m)}(x)]$$

to the kernel system

$$[K(x,y);\ K^{(1)}(x), \cdots, K^{(m)}(x)],$$

satisfying the relations

$$Q(x,y) = K(x,y) + \sum_{i=1}^{m} K^{(i)}(x)\,Q(\xi_i, y) + \int_a^b K(x,s)\,Q(s,y)\,ds,$$

$$Q(x,y) = K(x,y) + \sum_{i=1}^{m} Q^{(i)}(x)\,K(\xi_i, y) + \int_a^b Q(x,s)\,K(s,y)\,ds,$$

(12)

$$Q^{(p)}(x) = K^{(p)}(x) + \sum_{i=1}^{m} K^{(i)}(x)\,Q^{(p)}(\xi_i) + \int_a^b K(x,s)\,Q^{(p)}(s)\,ds,$$

$$Q^{(p)}(x) = K^{(p)}(x) + \sum_{i=1}^{m} Q^{(i)}(x)\,K^{(p)}(\xi_i) + \int_a^b Q(x,s)\,K^{(p)}(s)\,ds \qquad (p = 1, 2, \cdots, m).$$

Furthermore, (2) *possesses one and only one solution*:

$$u(x) = f(x) + \sum_{i=1}^{m} Q^{(i)}(x) f(\xi_i) + \int_a^b Q(x,s) f(s)\,ds; \tag{11}$$

and (2′) *possesses one and only one solution*:

$$v(y) = g(y) + \sum_{i=1}^{m} g^{(i)}\,Q(\xi_i, y) + \int_a^b g(s)\,Q(s,y)\,ds,$$

(11′)

$$v^{(p)} = g^{(p)} + \sum_{i=1}^{m} g^{(i)}\,Q^{(p)}(\xi_i) + \int_a^b g(s)\,Q^{(p)}(s)\,ds \qquad (p = 1, 2, \cdots, m).$$

7. We proceed now to study the case in which (1), (1′) possess non-trivial solutions. Let a complete set of linearly independent solutions of (1) be

$$u_1(x), \cdots, u_\nu(x).$$

By following out the form of proof used ordinarily in replacing a set of functions by a normal orthogonal set, we see that by proper linear combination the set of solutions may be made to satisfy the conditions:

$$\int_a^b u_\gamma(s)\,u_\rho(s)\,ds + \sum_{i=1}^{m} u_\gamma(\xi_i)\,u_\rho(\xi_i) = \begin{cases} 0, & \rho \neq \gamma. \\ 1, & \rho = \gamma. \end{cases} \tag{13}$$

Similarly, it is possible so to choose the ν linearly independent solutions of (1′):

$$\begin{aligned} &[v_1(y);\ v_1^{(1)},\ \cdots,\ v_1^{(m)}], \\ &[v_2(y);\ v_2^{(1)},\ \cdots,\ v_2^{(m)}], \\ &\cdots\cdots\cdots\cdots\cdots \\ &[v_\nu(y);\ v_\nu^{(1)},\ \cdots,\ v_\nu^{(m)}], \end{aligned}$$

that

$$\int_a^b v_\gamma(s)\,v_\rho(s)\,ds + \sum_{i=1}^{m} v_\gamma^{(i)}\,v_\rho^{(i)} = \begin{cases} 0, & \rho \neq \gamma. \\ 1, & \rho = \gamma. \end{cases} \tag{13′}$$

THEOREM IV.* *Corresponding to any complete set of linearly independent solutions $u_\gamma(x)$ $(\gamma = 1, 2, \cdots, \nu)$ of (1), it is possible to select a set of collections of the form*

$$[U_\gamma(y);\ U_\gamma^{(1)},\ \cdots,\ U_\gamma^{(m)}] \qquad (\gamma = 1, 2, \cdots, \nu)$$

each consisting of one continuous function and m constants, such that

$$\int_a^b u_\gamma(s)\,U_\rho(s)\,ds + \sum_{i=1}^{m} u_\gamma(\xi_i)\,U_\rho^{(i)} = \begin{cases} 0, & \rho \neq \gamma. \\ 1, & \rho = \gamma. \end{cases} \tag{14}$$

Corresponding to any complete set of linearly independent solutions

$$[v_\gamma(y);\ v_\gamma^{(1)},\ \cdots,\ v_\gamma^{(m)}] \qquad (\gamma = 1, 2, \cdots, \nu)$$

of (1′), it is possible to select a set of continuous functions $V_\gamma(x)$ $(\gamma = 1, 2, \cdots, \nu)$, such that

$$\int_a^b V_\gamma(s)\,v_\rho(s)\,ds + \sum_{i=1}^{m} V_\gamma(\xi_i)\,v_\rho^{(i)} = \begin{cases} 0, & \rho \neq \gamma. \\ 1, & \rho = \gamma. \end{cases} \tag{14′}$$

It is sufficient to show that the theorem holds for a single complete set of solutions of each equation; for to any linear transformation of the functions $u_\gamma(x)$ we may evidently choose an appropriate linear transformation of the sets $[U_\gamma(y);\ U_\gamma^{(1)}, \cdots, U_\gamma^{(m)}]$, so that the relations (14) remain unaltered; and to any linear transformation of the sets $[v_\gamma(y);\ v_\gamma^{(1)}, \cdots, v_\gamma^{(m)}]$ we may choose a linear transformation of the functions $V_\gamma(x)$ so that the relations

* The conditions (13), (13′), corresponding to orthogonality, are not sufficient for the completion of the theory. We therefore establish in Theorem IV certain conditions analogous to "biorthogonality" of ordinary functions.

(14′) remain unaltered. We therefore assume that the solutions have been so selected as to satisfy (13), (13′).

Slight consideration shows that the sets of U's may then be selected in a large variety of ways; for the proof of the theorem it will suffice to assign one choice satisfying the conditions. This is accomplished by taking

$$U_\rho(y) = u_\rho(y), \qquad U_\rho^{(p)} = u_\rho(\xi_p). \tag{15}$$

Similarly for the second part of the theorem take

$$V_\rho(x) = \sum_{i=1}^{m} K^{(i)}(x)\, v_\rho^{(i)} + \int_a^b K(x,s)\, v_\rho(s)\, ds. \tag{15′}$$

8. THEOREM V. *There exists a pseudo-resolvent system*

$$[Q(x,y);\ Q^{(1)}(x),\ \cdots,\ Q^{(m)}(x)]$$

to the kernel system

$$[K(x,y);\ K^{(1)}(x),\ \cdots,\ K^{(m)}(x)],$$

such that

$$\begin{aligned}
Q(x,y) &= K(x,y) + \sum_{i=1}^{m} K^{(i)}(x)\, Q(\xi_i, y) \\
&\qquad + \int_a^b K(x,s)\, Q(s,y)\, ds - \sum_{\gamma=1}^{\nu} V_\gamma(x)\, v_\gamma(y), \\
Q(x,y) &= K(x,y) + \sum_{i=1}^{m} Q^{(i)}(x)\, K(\xi_i, y) \\
&\qquad + \int_a^b Q(x,s)\, K(s,y)\, ds - \sum_{\gamma=1}^{\nu} u_\gamma(x)\, U_\gamma(y), \\
Q^{(p)}(x) &= K^{(p)}(x) + \sum_{i=1}^{m} K^{(i)}(x)\, Q^{(p)}(\xi_i) \\
&\qquad + \int_a^b K(x,s)\, Q^{(p)}(s)\, ds - \sum_{\gamma=1}^{\nu} V_\gamma(x)\, v_\gamma^{(p)}, \\
Q^{(p)}(x) &= K^{(p)}(x) + \sum_{i=1}^{m} Q^{(i)}(x)\, K^{(p)}(\xi_i) \\
&\qquad + \int_a^b Q(x,s)\, K^{(p)}(s)\, ds - \sum_{\gamma=1}^{\nu} u_\gamma(x)\, U_\gamma^{(p)} \\
&\qquad\qquad (p = 1, 2, \cdots, m).
\end{aligned} \tag{16}$$

The proof is similar to that used by the author in the case of pure integral equations.* Define the new kernel system:

* Loc. cit., p. 408.

$$L(x,y) = K(x,y) - \sum_{\gamma=1}^{\nu} V_\gamma(x)\, U_\gamma(y),$$

(17)

$$L^{(p)}(x) = K^{(p)}(x) - \sum_{\gamma=1}^{\nu} V_\gamma(x)\, U_\gamma^{(p)} \qquad (p = 1, 2, \cdots, m).$$

By (17), (1), (1'), (14), (14'), we have the identities:

$$\int_a^b L(x,s)\, u_\rho(s)\, ds + \sum_{i=1}^{m} L^{(i)}(x)\, u_\rho(\xi_i) = u_\rho(x) - V_\rho(x) \qquad (\rho = 1, 2, \cdots, \nu); \tag{18}$$

$$\int_a^b v_\rho(s)\, L(s,y)\, ds + \sum_{i=1}^{m} v_\rho^{(i)}\, L(\xi_i, y) = v_\rho(y) - U_\rho(y),$$

(18')

$$\int_a^b v_\rho(s)\, L^{(p)}(s)\, ds + \sum_{i=1}^{m} v_\rho^{(i)}\, L^{(p)}(\xi_i) = v_\rho^{(p)} - U_\rho^{(p)} \qquad (\rho = 1, 2, \cdots, \nu;\ p = 1, 2, \cdots, m).$$

An equation of the same form as (1') with the kernel system (17):

$$u(x) = \sum_{i=1}^{m} L^{(i)}(x)\, u(\xi_i) + \int_a^b L(x,s)\, u(s)\, ds$$

has no non-trivial solution. For from this equation follows by (18'), on applying to both sides the operation indicated by

$$\int_a^b v_\rho(s)\, u(s)\, ds + \sum_{i=1}^{m} v_\rho^{(i)}\, u(\xi_i),$$

the relation

$$\int_a^b U_\rho(s)\, u(s)\, ds + \sum_{i=1}^{m} U_\rho^{(i)}\, u(\xi_i) = 0 \qquad (\rho = 1, 2, \cdots, \nu). \tag{19}$$

Substituting in the integral equation the values of the new kernel system and simplifying by (19), we find that

$$u(x) = \sum_{i=1}^{m} K^{(i)}(x)\, u(\xi_i) + \int_a^b K(x,s)\, u(s)\, ds.$$

Hence $u(x)$ is a linear combination of $u_1(x), \cdots, u_\nu(x)$:

$$u(x) = c_1 u_1(x) + \cdots + c_\nu u_\nu(x).$$

By (14),

$$\int_a^b U_\rho(s)\, u(s)\, ds + \sum_{i=1}^{m} U_\rho^{(i)}\, u(\xi_i) = c_\rho \qquad (\rho = 1, 2, \cdots, \nu),$$

and by (19),

$$c_\rho = 0 \qquad (\rho = 1, 2, \cdots, \nu).$$

Thus $u(x)$ is identically zero, as we wished to show.

Since there are no non-trivial solutions for the kernel system

$$[L(x,y);\ L^{(1)}(x),\ \cdots,\ L^{(m)}(x)],$$

there exists a resolvent system

$$[Q(x,y);\ Q^{(1)}(x),\ \cdots,\ Q^{(m)}(x)],$$

such that

$$\begin{aligned}
Q(x,y) &= L(x,y) + \sum_{i=1}^{m} L^{(i)}(x)\,Q(\xi_i,y) + \int_a^b L(x,s)\,Q(s,y)\,ds,\\
Q(x,y) &= L(x,y) + \sum_{i=1}^{m} Q^{(i)}(x)\,L(\xi_i,y) + \int_a^b Q(x,s)\,L(s,y)\,ds,\\
Q^{(p)}(x) &= L^{(p)}(x) + \sum_{i=1}^{m} L^{(i)}(x)\,Q^{(p)}(\xi_i) + \int_a^b L(x,s)\,Q^{(p)}(s)\,ds,\\
Q^{(p)}(x) &= L^{(p)}(x) + \sum_{i=1}^{m} Q^{(i)}(x)\,L^{(p)}(\xi_i) + \int_a^b Q(x,s)\,L^{(p)}(s)\,ds\\
&\qquad (p = 1, 2, \cdots, m).
\end{aligned} \tag{20}$$

To the first equation apply the operation indicated by

$$\int_a^b v_\rho(s)\,Q(s,y)\,ds + \sum_{i=1}^{m} v_\rho^{(i)}\,Q(\xi_i,y);$$

after simplifying by (18′), we find

$$\int_a^b U_\rho(s)\,Q(s,y)\,ds + \sum_{i=1}^{m} U^{(i)}\,Q(\xi_i,y) = v_\rho(y) - U_\rho(y) \qquad (\rho = 1, 2, \cdots, \nu). \tag{21}$$

Writing out the values of the L's in the first equation of (20), and simplifying by (21), we have the first equation of (16). An entirely analogous set of operations on the third equation of (20) yields the third equation of (16).

Again, operate on the second and fourth equations of (20) as indicated by

$$\int_a^b Q(x,s)\,u_\rho(s)\,ds + \sum_{i=1}^{m} Q^{(i)}(x)\,u_\rho(\xi_i);$$

simplifying by (18), we have

$$\int_a^b Q(x,s)\,V_\rho(s)\,ds + \sum_{i=1}^{m} Q^{(i)}(x)\,V_\rho(\xi_i) = u_\rho(x) - U_\rho(x) \qquad (\rho = 1, 2, \cdots, \nu);$$

substituting this result in the forms obtained by writing out the values of the L's in the second and fourth equations of (20), we have the second and fourth equations of (16).

9. By means of the pseudo-resolvent system we can complete the theory of the solution in the non-homogeneous case, as follows:

THEOREM VI. *In order that* (2) *possess a solution it is necessary and sufficient that*

$$\int_a^b v_\gamma(s) f(s)\, ds + \sum_{i=1}^{m} v_\gamma^{(i)} f(\xi_i) = 0 \qquad (\gamma = 1, 2, \cdots, \nu).$$

If this condition is satisfied, every solution may be written in the form:

$$u(x) = f(x) + \sum_{i=1}^{m} Q^{(i)}(x) f(\xi_i) + \int_a^b Q(x, s) f(s)\, ds + \sum_{\gamma=1}^{\nu} c_\gamma u_\gamma(x).$$

THEOREM VI′. *In order that* (2′) *possess a solution it is necessary and sufficient that*

$$\int_a^b g(s)\, u_\gamma(s)\, ds + \sum_{i=1}^{m} g^{(i)}\, u_\gamma(\xi_i) = 0 \quad (\gamma = 1, 2, \cdots, \nu).$$

If this condition is satisfied, every solution may be written in the form:

$$v(y) = g(y) + \sum_{i=1}^{m} g^{(i)}\, Q(\xi_i, y) + \int_a^b g(s) Q(s, y)\, ds + \sum_{\gamma=1}^{\nu} c_\gamma v_\gamma(y),$$

$$v^{(p)} = g^{(p)} + \sum_{i=1}^{m} g^{(i)}\, Q^{(p)}(\xi_i) + \int_a^b g(s) Q^{(p)}(s)\, ds + \sum_{\gamma=1}^{\nu} c_\gamma v_\gamma^{(p)}$$
$$(p = 1, 2, \cdots, m).$$

The proof of these two theorems follows the same lines as that of the corresponding developments in the case of pure equations.*

* W. A. Hurwitz, loc. cit., p. 409.

CORNELL UNIVERSITY,
July, 1914.

PRIME POWER GROUPS IN WHICH EVERY COMMUTATOR OF PRIME ORDER IS INVARIANT*

BY

WILLIAM BENJAMIN FITE

In this paper I represent by G a group of order p^m (p an odd prime) and class k; by H_j the jth central of† G; and by G' the first cogredient of G. The term "ith commutator" is used in the sense explained in an earlier paper.‡

If the central of G' is cyclic, H_3 must contain an operation s that is not contained in H_2 whose pth power is contained in H_1, since otherwise the quotient group H_3/H_1, which is the second central of G', would contain only one subgroup of order p, and would therefore be cyclic.§ But this is impossible.‖ Also G must contain an operation, as A, such that $A^{-1}sA = st$, where t is contained in H_2, but not in H_1. Moreover if $s^{-1}ts = th$, h is invariant in G and $s^{-p}ts^p = th^p = t$, since s^p is invariant in G. Hence the order of h does not exceed p. It follows that¶ $(st)^p = s^p t^p h^{p(p-1)/2} = s^p t^p$. Hence $A^{-1}s^pA = (st)^p = s^p t^p$, and $t^p = 1$. This proves that *if G' has a cyclic central, H_2 must contain a commutator of order p that is not invariant in G.*

If now any cogredient of G, as the ith $(1 < i < k-1)$, contains a cyclic central, H_{i+1}/H_i must be cyclic; and it follows from what has just been proved that there must be contained in H_{i+1} a commutator t_{i+1} which is not contained in H_i, but whose pth power is contained in H_{i-1}. There must then be some operation of G, as A_{i+1}, such that

$$A_{i+1}^{-1} t_{i+1} A_{i+1} = t_{i+1} t_i,$$

where t_i is a second commutator contained in H_i, but not in H_{i-1}, and t_i^p is contained in H_{i-2}. If $i > 2$, there must be an operation A_i such that

$$A_i^{-1} t_i A_i = t_i t_{i-1},$$

* Presented to the Society at the Providence meeting, September 9, 1914.

† This notation is that of Burnside's *Theory of Groups of Finite Order*, first edition, p. 62, and not that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

‡ Fite, these Transactions, vol. 7 (1906), p. 61.

§ Burnside, loc. cit., second edition, p. 131.

‖ Fite, these Transactions, vol. 15 (1914), p. 48.

¶ Fite, these Transactions, vol. 3 (1902), p. 337.

where t_{i-1} is a third commutator contained in H_{i-1}, but not in H_{i-2}, and t_{i-1}^p is contained in H_{i-3}. We can continue this argument until we arrive at an ith commutator of order p that is contained in H_2 but not in H_1. Hence *if any cogredient of G, as the ith, has a cyclic central, H_2 must contain an ith commutator of order p that is not invariant in G.*

If every ith ($i < k-1$) commutator of order p is invariant in G, as is the case, for example, when the ith commutator subgroup is cyclic, neither the ith cogredient nor any succeeding cogredient can have a cyclic central. Moreover there is no operation of H_{i+2} that is not contained in H_{i+1} whose pth power is contained in H_i. For if s were such an operation, there would be an operation, A, of G such that

$$A^{-1} sA = st_{i+1},$$

where t_{i+1} is a commutator contained in H_{i+1}, but not in H_i. It follows from this that

$$A^{-1} s^p A = s^p t_{i+1}^p t,$$

where t is contained in H_{i-1}, as may be seen by applying an argument similar to the one on page 134 to the corresponding operations of the quotient group G/H_{i-1}. Hence, since s^p is by hypothesis contained in H_i, t_{i+1}^p must be contained in H_{i-1}. But the preceding argument shows that this is possible only in case G contains a non-invariant ith commutator of order p.

Since every $(i+1)$th commutator is also an ith commutator,* there can be no operation of H_{i+j} ($j > 2$) that is not contained in H_{i+j-1} whose pth power is contained in H_{i+j-2}. Hence the order of every operation of G that is not contained in H_{k-1} must be at least p^{k-i}.

If we suppose that every ith commutator that is contained in H_2 and whose order does not exceed p^r ($r > 0$) is invariant in G, we shall see that the facts just mentioned are particular phases of somewhat more general ones.

For if an operation s of H_{i+2} that is not contained in H_{i+1} were such that its p^rth power is contained in H_i, there would be an operation A of G such that

$$A^{-1} sA = st_{i+1},$$

where t_{i+1} is contained in H_{i+1}, but not in H_i. Moreover

$$A^{-1} s^{p^r} A = s^{p^r} t_{i+1}^{p^r} t,$$

where t is contained in H_{i-1}. Since s^{p^r} is contained in H_i, $t_{i+1}^{p^r}$ must be contained in H_{i-1}. If $i = 1$ this would be contrary to the hypothesis, since the order of the non-invariant first commutator t_{i+1} would be equal to, or less than, p^r. If $i > 1$, we can apply a similar argument to t_{i+1} and continue the process until we arrive at a non-invariant ith commutator that is contained

* Fite, these Transactions, vol. 7 (1906), p. 61.

in H_2 and whose order does not exceed p^r. But by hypothesis there is no such commutator. Hence *if every ith commutator of G that is contained in H_2 and whose order does not exceed p^r is invariant in G, the p^rth power of an operation of H_{i+2} that is not contained in H_{i+1} cannot be contained in H_i.*

It follows that the order of every operation of H_{i+j} $(j \geqq 2)$ that is not contained in H_{i+j-1} must be greater than $p^{r(j-1)}$, and that every operation not contained in H_{k-1} must be of order greater than $p^{r(k-i-1)}$. In particular, if every commutator whose order does not exceed p^r is invariant, the order of every operation that is not contained in H_{k-1} must be greater than $p^{r(k-2)}$.

The following statement follows from the preceding general discussion as an important special case: *If every operation of H_2 whose order does not exceed p^r is invariant in G, every operation of G whose order does not exceed p^r is invariant.*

If every ith commutator of order p that is contained in H_2 is invariant in G, no cogredient after the ith one can have a central that is generated by more independent generators than the central of the immediately preceding cogredient. This can be seen by comparing first the number of independent generators of the central of a group L of order a power of p with the number of independent generators of the central of the first cogredient of L. Let L_j and L' have the same significance with reference to L that H_j and G' respectively have with reference to G.

If the pth power of some operation of L_2 that is not contained in L_1 is equal to the pth power of some operation of L_1, there must be contained in L_2 an operation of order p that is not contained in L_1. If there is no such operation in L_2, the pth power of an operation of L_2 that corresponds to an operation of order p of L' must be some one of the $p^{a-n_1}(p^{n_1}-1)$ operations of L_1 that are not pth powers of operations of L_1. Here p^a and n_1 represent respectively the order of L_1 and the number of its independent generators. If the central of L' is generated by n independent generators, L_2 contains $p^a(p^n-1)$ operations that correspond to operations of order p of the central of L'. Moreover these operations can be divided into sets of p^{n_1} operations each such that the operations of any set have the same pth power, while two operations of different sets have different pth powers. This follows from the fact that L_1 contains p^{n_1} operations whose orders do not exceed p and the assumption that every operation of L_2 not contained in L_1 is of order greater than p. Hence L_1 must contain $p^{a-n_1}(p^n-1)$ operations that are pth powers of operations of L_2, but not pth powers of operations of L_1. This would be impossible if $n > n_1$.

If now the central of the $(i+j)$th cogredient of $G(j \geqq 1)$—that is, the central of G/H_{i+j}—were generated by n independent generators, while the central of G/H_{i+j-1} is generated by n_1 independent generators, where $n > n_1$,

we should have the case just considered with G/H_{i+j-1} for L. Hence the pth power of some operation of H_{i+j+1} not contained in H_{i+j} would be contained in H_{i+j-1}. But this is impossible since every ith commutator of order p that is contained in H_2 is invariant in G.

Suppose now that every commutator of G that is contained in H_2 and whose order does not exceed p^r is invariant in G, and that G contains an invariant commutator of this order, but none of higher order. Then the p^rth power of any operation of any central after the first one is contained in the preceding one. Moreover in any cogredient of G the commutators in the second central whose orders do not exceed the order of every invariant commutator of this cogredient must themselves be invariant in this cogredient.

If H_{k-1} is abelian let s be any one of its operations and A any operation of G not contained in H_{k-1}. Then

$$s^{-1}As = At,$$

where t is contained in, say, H_j, but not in H_{j-1}. Suppose that

$$A^{-1}tA = tt_1.$$

Then t_1 is contained in H_{j-1}. If now A', t', and t_1' are the operations of G/H_{j-2} that correspond respectively to A, t, and t_1, then

$$(A't')^{p^r} = A'^{p^r} t'^{p^r} t_1'^{p^r(p^r-1)/2},$$

since t_1' is commutative with both A' and t'. Hence

$$s^{-1}A^{p^r}s = A^{p^r} t^{p^r} t_1^{p^r(p^r-1)/2} t_2,$$

where t_2 is contained in H_{j-2}. But A^{p^r} is contained in H_{k-1} and is therefore commutative with s. Hence

$$t^{p^r} t_1^{p^r(p^r-1)/2} t_2 = 1.$$

This implies that $t^{p^r} = 1$ since $t_1^{p^r(p^r-1)/2} t_2$ is contained in H_{j-2} and t^{p^r} cannot be contained in H_{j-2} unless this is composed of the identity alone. Since then the order of the commutator t does not exceed p^r, t must be invariant in G. Hence no operation of H_{k-1} can give a non-invariant commutator and H_{k-1} must be contained in H_2. This requires that $k \leqq 3$.

Suppose now that H_{k-1} is of class $k_1 (> 1)$ and assume that $k \leqq 2j + 1$ when H_{k-1} is of class j and $j < k_1$. We have just seen that this assumption is valid when H_{k-1} is abelian. Every $(k_1 - 1)$th commutator of H_{k-1} is invariant in H_{k-1}. If we take any such commutator for the s of the argument just used for the case $j = 1$, we shall see that this commutator must be contained in H_2. Hence the class of the corresponding central of G/H_2 is

less than, or equal to,* $k_1 - 1$. But G/H_2 is of class $k-2$, and therefore in accordance with our assumption $k - 2 \leqq 2(k_1 - 1) + 1$, or $k \leqq 2k_1 + 1$.

In general, if H_{k-l} $(l > 1)$ is abelian, let s be any one of its operations and A any operation of G not contained in H_{k-l}. Then

$$A^{-1} sA = st,$$

where t is contained in, say, H_j, but not in H_{j-1}. If

$$A^{-1} tA = tt_1$$

and

$$A^{-1} t_n A = t_n t_{n+1} \qquad (n = 1, 2, \cdots),$$

then

$$A^{-p^{lr}} sA^{p^{lr}} = st^{p^{lr}} t_1^{p^{lr}(p^{lr}-1)/2} \cdots t_n^x \cdots t_{n_1}^y,$$

when x equals the coefficient of the $(n+2)$th term in the expansion of the p^{lr}th power of a binomial and $y = 1$ or the coefficient of the $(n_1 + 2)$th term in this expansion, according as $n_1 = p^{lr} - 1$ or as t_{n_1} is the first of this series of commutators with which A is commutative. Now H_{j-n} contains t_n and H_{j-l-1} contains

$$t_1^{p^{lr}(p^{lr}-1)/2} \cdots t_n^x \cdots t_{n_1}^y,$$

but not $t^{p^{lr}}$, unless H_{j-l} is composed of the identity alone. But $A^{p^{lr}}$ is commutative with s. Hence $t^{p^{lr}} = 1$. It follows from this that H_{k-l} must be contained in H_{l+1}, and $k - l \leqq l + 1$, or $k \leqq 2l + 1$. Suppose now that H_{k-l} is of class $k_l (> 1)$ and assume that $k \leqq jl + j + l$ when H_{k-l} is of class j and $j < k_l$. We have just seen that this assumption is valid when H_{k-l} is abelian. Every $(k_l - 1)$th commutator of H_{k-l} is invariant in H_{k-l}, and can therefore give in G only commutators whose orders do not exceed p^{lr}. Every such $(k_l - 1)$th commutator must therefore be contained in H_{l+1}. Hence the class of the central of G/H_{l+1} that corresponds to H_{k-l} is less than, or equal to, $k_l - 1$. But G/H_{l+1} is of class $k - l - 1$, and therefore in accordance with our assumption $k - l - 1 \leqq (k_l - 1)l + k_l - 1 + l$, or $k \leqq k_l l + k_l + l$. Hence

If G is of order p^m (p an odd prime) class k and if every commutator of G that is contained in H_2 and whose order does not exceed the highest order of any invariant commutator is itself an invariant commutator, then H_{k-l} is of class k_l, where $k \leqq k_l l + k_l + l$.

Groups with cyclic commutator subgroups and of order p^m are included in the category considered here.

COLUMBIA UNIVERSITY,
November, 1914.

* Fite, loc. cit.

ON THE ORDER OF PRIMITIVE GROUPS, II*

BY

W. A. MANNING

As early as 1873 Jordan proved that *a non-alternating primitive group cannot contain a substitution of degree and order p (p a prime) if its degree exceeds $p+2$.* On page 176 of the first volume of the Bulletin of the Mathematical Society of France he explicitly stated this theorem, as having been proved on page 42 of the same volume. There one finds, *"Let p be an odd prime; a group of degree $p+k$ cannot be more than k times transitive unless it contains the alternating group."* This of course is seen to cover the theorem stated, if one has in mind (from page 664 of the *Traité des Substitutions*, 1870) *"If a group contains a circular substitution of prime order p it is $n-p+1$ times transitive."* It is easy to infer that the order of a non-alternating primitive group of degree n cannot exceed $n!/(2\cdot 3\cdots p)$, where $2, 3, \cdots, p$ are the distinct primes less than $n-2$. For if p^α is the highest power of p in $n!$ a group of order p^α and of degree not greater than n certainly contains substitutions of degree and order p. Even so we have a great improvement on the theorem that asserts a limit of the same form but imposes upon p the greater restriction $p < 2n/3$.†

In the paper‡ bearing the same title as the present one, and of which this is a continuation, was proved

THEOREM X. *Let q be an integer greater than unity and less than 5; p any prime greater than $q+1$; then the degree of a primitive group which contains a substitution of order p that displaces pq letters (not including the alternating group) cannot exceed $pq+q$. When p is equal to $q+1$, the degree cannot exceed $pq+q+1$.*

By means of this theorem and our knowledge of the primitive groups of class less than 14§ it is possible to further limit the order of a primitive group of given degree. The desired connection is established by means of

THEOREM XI. *Let p^α, p^β be the highest powers of p that divide $\lfloor n$ and*

* Presented to the Society (San Francisco Section), Feb. 27, 1909.

† Burnside, *Theory of Groups*, 1st edition, p. 199; 2d edition, p. 207.

‡ These Transactions, vol. 10 (1909), p. 247.

§ *The Primitive Groups of Class Twelve*, American Journal of Mathematics, vol. 35 (1913), p. 229.

pq respectively, $n > pq$. Let G be a group of degree n that contains no substitution of order p on so few as q cycles. Then the Sylow subgroup of G corresponding to the prime p is not of higher order than $p^{\alpha-\beta}$.

Suppose the order of the Sylow subgroup of G is $p^{\alpha-\beta+k}$. The Sylow subgroup of the symmetric group of degree n which includes this group of order $p^{\alpha-\beta+k}$ contains also a subgroup of order p^{β}, the Sylow subgroup of the symmetric group of degree pq. Since these two subgroups, of orders $p^{\alpha-\beta+k}$ and p^{β}, have no substitution in common, k is necessarily zero.

Then if the order of a primitive group G of degree n does not divide $n!/(2^{10}3^5)$ its class is less than 13, and by theorem X, the order of G (of class greater than 12) must divide $n!/(5^4 \cdot 7^4 \cdots p_4^4 \cdots p_3^3 \cdots p_2^2 \cdots p_1)$, where $5, 7, \cdots p_1$ are all the distinct primes greater than 3 and less than $n-2$, where those primes less than $n/2-1$ are squared, those less than $n/3-1$ are cubed, those less than $n/4-1$ are raised to the fourth power, unless G is of degree 25 and contains a transitive Sylow subgroup of order 125.*

Jordan also proved that *if a primitive group G contains a substitution of prime order p on 5 cycles, $p > 5$, and does not include the alternating group, its degree cannot exceed $5p+6$.* If this theorem is used, and a proof of it will presently be given, we may formulate the inclusive theorem:

THEOREM XII. *The order of a primitive group (G) of degree n and of class greater than 12 divides $n!/(2^{10} P_1 P_2 P_3 P_4 P_5)$, where P_1 is the product of all the odd primes less than $n-2$, P_2 the product of those less than $n/2-1$, P_3 the product of those less than $n/3-1$, P_4 the product of those less than $n/4-1$, and P_5 the product of those less than $(n-1)/5-1$, unless (1) G, of degree 25, contains a substitution of order 5 on 4 cycles, or (2) G, of degree not greater than 45,† contains a substitution of order 5 on 5 cycles.*

The exact determination of these exceptions is not here attempted. The author prefers to leave the question open until he has had opportunity to complete the examination of the groups of classes 14 and 15. The remainder of this paper will be devoted to a proof of that part of Jordan's theorem that remains as yet without a published proof,—that relating to *the primitive groups which contain a substitution of prime order p $(p > 5)$ on 5 cycles.*

From theorem IV of the preceding paper it follows that the primitive group G in question contains a transitive subgroup H generated by substitutions of order p and of degree pq. These generators $A, B, \cdots$ are of such a nature that if an intransitive subgroup I_1 generated by a certain number of them taken in order be fixed upon, then the next substitution in the series connects at least two sets of intransitivity of I_1 and has in no cycle more than

---

* *On the Order of Primitive Groups*, these Transactions, vol. 10 (1909), pp. 255 and 256.

† For this limit 45, see these Transactions, vol. 12 (1911), p. 375.

one letter not already displaced by I_1. Hence the degree of H does not exceed $pq + (q-1)q$.

When H is an imprimitive group, generated by substitutions of order p and degree pq, the systems of imprimitivity are permuted by these substitutions of order p, so that *the number of letters in any system divides* q. In the present case q is 5, so that imprimitive systems of 5 letters each are permuted according to a primitive group of degree $p+k$, $0 \leqq k \leqq 4$. The new letters introduced by B, or by C, $\cdots$ form a system of imprimitivity. Then if $\{A, B\}$ is not of degree $5p$, it is of degree $5p+5$ and causes the group* J to be of class 5 or less. Now the head of H, when it is not the identity, has transitive constituents of degree 5. Then the order of H divides $(5!)^{p+k}((p+k)!)$. This number is not divisible by p^2. Hence all subgroups of H of order p are Sylow subgroups. Then J is a transitive group, and must have imprimitive systems of 5 letters each. When k is greater than unity there are substitutions of class less than 5 in J and they can permute no systems of 5 letters; therefore $J(k>1)$ has an intransitive subgroup leaving fixed each of the k systems. Each constituent of this head is of degree 5 and of order greater than 5. If the letters of a substitution in J of degree 4 or less all belong to the same system, the head is a direct product. If they belong to different systems, each constituent is a symmetric group of degree 5. But we recall that J is a transitive representation on $5k$ ($k = 2, 3, 4$) letters of the direct product of a cyclic group and a subgroup of the symmetric group on 5 symbols. The symmetric group is included among its own subgroups. Then J can only be of class 4 with $k = 2$, when it is a simple isomorphism between two symmetric groups permuted by an invariant substitution of order 2. Since systems of imprimitivity of H may be chosen in but one way, $H(k=2)$ must be found in a primitive group of degree $5p+10$ or $5p+11$. However J cannot be written as a non-regular group of degree 11. After having studied the simply transitive primitive groups of degree $5p+10$ we shall return to the imprimitive group H of that degree and show that it cannot be contained in a primitive group of degree $5p+10$.

In case H is a multiply transitive group of degree greater than $5p$, H_1, the subgroup leaving one letter fixed, cannot be primitive, for then the subgroups of order p and degree $5p$ in H_1 would generate a transitive group. H_1 is therefore imprimitive and the substitutions of order p and degree $5p$ generate an intransitive group with the same number $p+k$ of letters in each constituent. Then H_1 does not involve the subgroup $\{A, B\}$, which it certainly would, were its degree as great as $5p+5$. Then H_1 is of degree $5p$, and H, when doubly transitive, is of degree not greater than $5p+1$.

* I is the largest subgroup of the transitive group under discussion in which $\{A\}$ is invariant, and J is the constituent (or constituents) of I in the letters left fixed by A.

Let us now assume that H is a simply transitive primitive group. The subgroup H_1 that leaves one letter fixed is intransitive and displaces all but one of the letters permuted by the group H. Another theorem for which we are indebted to Jordan is that if a given prime p divides the order of one transitive constituent of the subgroup leaving fixed one letter of a simply transitive primitive group, then does p divide the order of every constituent of that subgroup. Hence H_1 *has no transitive constituent of degree less than* p.

LEMMA I. *If a constituent of* H_1 *is alternating or symmetric, the degree of* H *is at most* $5p + 6$ *and* p *is* 7. *In no case is* H *of higher degree than* $5p + 10$, *nor does* H *contain a subgroup of order* p^2.

If a substitution of order 3 on 15 letters is present in H, it has elsewhere been proved that the upper limit of the degree of H is 41.* This is possible only when p is 7, and then $41 = 5 \cdot 7 + 6$.

Let us make the assumption, to be removed later, that there is no substitution of order 5 and degree 25 in H.

If a constituent of maximum degree is alternating the remaining constituents are simply isomorphic to it. Now when n is greater than 6 the symmetric group of degree n is the holomorph of the alternating group of degree n, so that here, where the degree of any constituent of H_1 is at least 7, H_1 contains substitutions of order 3 on as many cycles as it has constituents. Hence a constituent of maximum degree cannot be an alternating or symmetric group. Let us see if any constituent of H_1 can include the alternating group of its degree.

Suppose that H_1 has just two constituents. If one constituent is alternating it is of lower degree than the other. Again, if the isomorphism is simple, the alternating group can only be of degree 7 or 8 ($p = 7$) and the other constituent is of degree $4 \cdot 7 + 2 = 30$ or $4 \cdot 7 = 28$ respectively, so that H_1 is of degree $5p + 2$ or $5p + 1$, cases which occasion no difficulty. Let the smaller constituent be alternating with the second constituent in an $(m, 1)$ isomorphism to it. The subgroup of order m cannot be transitive, since H is simply transitive. Then the larger constituent is imprimitive. If one substitution of order p in the larger constituent permutes systems, all the substitutions of order p permute systems and there are in this case no substitutions of order p in the intransitive head. Then the tail, isomorphic to an alternating group, permutes the systems accordingly. There are not more than four letters in a system, so that there is a substitution of order 5 and degree 25 in H_1. Now one substitution of order p in the larger constituent certainly permutes systems because at least one such substitution is in the tail of that constituent. The above applies *a fortiori* if the smaller constituent is symmetric. The conclusion is that neither of *two* constituents

* These Transactions, vol. 12 (1911), p. 375.

can be alternating or symmetric. It follows that the degree of H_1 is not in excess of $5p + 10$.

If p^2 divides the order of H_1, p can only be 7 and one of the two constituents is of degree $4p + 8$, an imprimitive group with $p + 2$ systems of 4 letters each. Such a group however does not admit the factor 49 in its order.

Let H_1 have three constituents, and suppose that one of them, not of maximum degree, is alternating. If a second constituent is of the same degree, it is also alternating. Since the larger constituent cannot be in simple isomorphism ($p \geqq 7$) to these groups, it would have an intransitive head, be imprimitive, and require a substitution of order 5 and degree 25 in H_1. Then both the remaining constituents are of higher degree than the alternating constituent and neither can be of degree less than $2p$. The alternating group is of degree $p + k$. In neither of the two groups of higher degree is the invariant subgroup the identity. Nor can both these constituents be imprimitive. For consider an imprimitive group of degree $2p + k_1$. If no substitution of order p permutes systems, two alternating groups in simple isomorphism are permuted by a substitution of order 2, which after all means that substitutions of order p permute systems of two letters each in all cases. Then H_1 contains a substitution of order 3 and of degree 15. Hence at least one of the larger constituents is primitive. It is evident that in this event p^2 does not divide the order of H_1 even when p is 7, because the order of a non-alternating primitive constituent of degree at most $2p + 2$ is not divisible by $p^2 = 49$. If the latter were divisible by p^2 it would contain a cycle of p letters. Now consider a non-alternating primitive constituent of degree $2p + k_1$. If the subgroup leaving one letter fixed is imprimitive, it either contains invariantly a simple isomorphism between two alternating groups each of degree greater than 6, or else permutes systems of two letters each according to an alternating group. But this constituent cannot be of so low a class as 6. Again, if the subgroup leaving one letter fixed is primitive, it is in a multiple isomorphism to an alternating group, and by applying the same analysis we must at last come to a subgroup that is a simple isomorphism between two alternating groups and is of class 6, or to an imprimitive subgroup that has systems of two letters permuted by an alternating group, also of class 6. Finally if the constituent in question is simply transitive its subgroup leaving one letter fixed is a simple isomorphism between two alternating groups, giving class 6 again. Hence when there are just three constituents in H_1 no one of them can be alternating (or symmetric). This again reduces the degree of H_1 to $5p + 10$ at most. The order of H_1 is not divisible by p^2.

Let there be four constituents in H_1. Now H_1 does not include the group $\{A, B, C\}$ and its degree is at most $5p + 9$. Three constituents are alter-

nating groups of equal degree $p + k$. Except for $p = 7$, with one constituent of degree 15 and the others of degree 8, and H_1 of degree $5p + 4$, the constituent of degree $2p + k_1$ cannot be in simple isomorphism to these alternating groups, nor can it have an invariant transitive head. If it is imprimitive with $p + k$ systems of two letters each, H_1 includes a substitution of degree 15 and order 3.

If there are five constituents, H_1 is of degree less than $5p + 5$.

Let us now remove the restriction upon H, that it contain no substitution of order 5 and degree 25. If such a substitution is present, H is not of higher degree than 45, and consequently $p = 7$. We return to the cases in the preceding paragraphs in which use was made of the presence of a substitution of order 5 in H.

Case when H_1 has just two constituents. The larger is imprimitive and in an $(m, 1)$ isomorphism to an alternating or symmetric group. Substitutions of order 11 in the imprimitive group cannot permute systems at the same time with substitutions of order 7. Then the alternating constituent is at most of degree 10. Hence the constituents of H_1 are of the degrees 7, 28 or 8, 32.

Case when H_1 has three constituents. The largest is imprimitive and there are two alternating constituents of equal degree less than 11 as before. Hence the degrees are 7, 7, 21 or 8, 8, 24 respectively.

In no case does 49 divide the order of H. This completes the proof of Lemma I.

We shall now prepare a list of the groups J of degree $6, \cdots, 10$ inclusive that are non-regular and with its aid prove

LEMMA II. *The transitive group $H = \{A, B, \cdots\}$ is in no case of higher degree than $5p + 5$.*

The principle employed in the construction of this list is that J is the direct product of a cyclic group and a subgroup of the symmetric group of order 120, the product being represented transitively on n letters, $n = 6, 7, \cdots, 10$.

Let h be the order of the cyclic group and kk' the order of the subgroup of the symmetric-5 group. The necessary and sufficient condition for the representation is that the subgroup (K) of the symmetric-5 have a subgroup (K') of order k' such that no subgroup of K' is invariant in K and that $kh = n$. Since $k' \neq 1$, the least value of k is 3, so that h is one of the numbers 1, 2, or 3. We arrange these groups in the descending order of their degrees.

There are 8 groups J of degree 10. In the first four $h = 1$.

(1) The symmetric G_{120}^5 gives an imprimitive J_{120}^{10}, in which J_1 (the subgroup leaving one letter fixed) is the G_{12}^4, here intransitive on two sets of letters. Each constituent of J_1 is doubly transitive. The class is 6.

(2) The G_{120}^{5} has an intransitive subgroup of order 12 with respect to which we have a primitive J_{120}^{10}. The J_1 is intransitive. One constituent is of order 12 and the other of order 6. This J is the group of the permutations of the ten products $ab, ac, \cdots$. It contains negative substitutions and is of class 6.

(3) The positive subgroup J_{60}^{10} is primitive and of class 8. J_1 is a $(1, 1)$ isomorphism between a regular and a non-regular symmetric-3 group.

(4) The metacyclic G_{20}^{5} is an imprimitive J_{20}^{10} of class 8.

In the remaining four groups of degree 10, $h = 2$.

(5) The direct product of the symmetric-5 and the cyclic-2 gives us an imprimitive $\pm J_{240}^{10}$. The J_1 is G_{24}^{4} on two sets of four letters each. The class is 4.

(6) The direct product of the alternating-5 and the cyclic-2 is an imprimitive $\pm J_{120}^{10}$. J_1 is G_{12}^{4} on two sets of four letters each. The class is 6.

(7) The direct product of the metacyclic G_{20}^{5} and a cyclic-2 is an imprimitive $\pm J_{40}^{10}$. The J_1 is the cyclic-4 on two sets of letters. The class is 8.

(8) The direct product of the semi-metacyclic G_{10}^{5} and the cyclic-2 is an imprimitive $\pm J_{20}^{10}$. It is of class 8.

The H_1 of a primitive group H of degree $5p + 10$ is an intransitive group of degree $5p + 9$, and has not more than four constituents since it contains $\{A, B\}$. No constituent is alternating. There cannot be just two constituents, for one of them would necessarily be of degree $4p + 8$, from which it would follow that J_1 is transitive. But on consulting the above list this is seen to be impossible. If there are three constituents of the degrees $3p + n$, $p + n', p + n''$, we must have $n = 6, n' = 2$, so that J_1 has a constituent of degree 6 along with one of degree 2. This our list forbids. If the partition of the degree of H_1 corresponding to its transitive sets is $2p + n, 2p + n'$, $p + n''$, then must $n = n' = 4$. The first two constituents are each imprimitive with $p + 2$ systems of two letters each. J is transitive of degree 10, J_1 is intransitive of degree 8, with two sets of four letters each. These constituents of J_1 are imprimitive, since J_1 must respect the systems of H_1. Furthermore each constituent of J_1 involves a transposition. But in none of the groups J^{10} as listed does J_1 involve an octic constituent. Finally let there be four constituents in H_1. Their degrees can only be $2p + 4, p + 2, p + 2$, $p + 1$, always bearing in mind that no constituent is alternating. Now if $\{A, B\}$ has constituents of degrees $2p + 2, p + 1, p + 1, p + 1, J$ involves a transposition, which we know is not possible, consequently the degree of the large constituent of $\{A, B\}$ is not greater than $2p + 1$. This leads to the contradiction that H_1 includes $\{A, B, C\}$.

We return to the hypothesis that H is an *imprimitive* group of degree $5p + 10$. It was shown that J in this case is (5) of our list. Since there is a

subgroup of degree $5p+5$, transitive in $p+1$ systems, the other five letters belong to one system, that is, systems of five letters can be chosen in only one way. Then H is included in a primitive group of degree $5p+10$ or $5p+11$. The latter is impossible since J cannot be of degree 11. Nor can a primitive group containing H be of degree $5p+10$. For if it is simply transitive we apply to it the reasoning employed in the case of a simply transitive primitive H of degree $5p+10$. Since J^{10} is in no case doubly transitive, this group cannot be doubly transitive. Hence H, if imprimitive, is not of higher degree than $5p+5$.

Next in order we have to consider the simply transitive primitive groups H of degree less than $5p+10$. In continuation of our list of possible groups J, we note that if $h=1$ or $h=2$, there is no transitive J, but if $h=3$ we have one group:

(9) The direct product of the symmetric group of order 6 and the cyclic-3 represented on 9 letters. This imprimitive $\pm J^9_{18}$ is of class 6.

In a simply transitive primitive group H of degree $5p+9$, H_1 is of degree $5p+8$ and has at most four constituents. No partition of $5p+8$ is consistent with a J_1 of order 2 and degree 6.

The groups J on 8 letters are:

(10) The G^4_{24} may be written as an imprimitive group of class 6.

(11) The direct product of the G^4_{24} and the cyclic-2 is an imprimitive J^8_{48} of class 4.

(12) The direct product of G^4_{12} and G^2_2. This J^8_{24} is of class 6.

(13) The direct product of G^4_8 and G^2_2. This J^8_{16} is of class 4.

Consider the intransitive H_1 of degree $5p+7$. There are at most 4 constituents. The (10), (11), (12) are not possible as groups J because they call for two constituents of degree $lp+3$ in H_1. Neither is (13) possible because two constituents of degree $lp+2$ do not permit us to build up the degree $5p+7$.

Let J be of degree 6. We represent

(14) G^5_{120} as a triply transitive $\pm J^6_{120}$ of class 4.

(15) G^5_{60} as a doubly transitive J^6_{60} of class 4.

(16) G^4_{24} as an imprimitive J^6_{24}. J_1 is axial and transitive.

(17) G^4_{24} as an imprimitive J^6_{24}. J_1 is cyclic and transitive.

(18) G^4_{12} as an imprimitive J^6_{12} of class 4.

(19) $G^3_6 \times G^2_2$ as an imprimitive J^6_{12} of class 4.

Now H is of degree $5p+6$. Since H_1 is intransitive the five letters of J_1 can not form a single transitive set. This bars (14) and (15). If in I (the largest subgroup of H in which $P=\{A\}$ is invariant) the five cycles of A were connected transitively, then would substitutions of order 5 certainly be found in it. Hence the cycles of A are not connected transitively by I. In (16), (17), and (18) the intransitive head of J is the axial group

$$1, \quad \alpha\beta \cdot \gamma\delta, \quad \alpha\beta \cdot \epsilon\zeta, \quad \gamma\delta \cdot \epsilon\zeta,$$

and when we adjoin $\alpha\beta \cdot \epsilon\zeta$ to the subgroup of J which leaves ϵ fixed we obtain an intransitive subgroup of J. Hence there is a substitution in I, not in H_1, which with H_1 generates an intransitive subgroup of H. But H_1 is maximal since H is primitive. This leaves only (19) to which the same reasoning applies, inasmuch as the invariant head of this diedral rotation group is $1, \alpha\beta \cdot \gamma\delta \cdot \epsilon\zeta$.

It has now been proved that G contains a transitive group $H = \{A, B, \cdots\}$ of degree not greater than $5p + 5$. If H is imprimitive its degree is $5p$ or $5p + 5$. Since in the latter case systems of five letters can be chosen in but one way, H, if contained in a primitive group of higher degree is found in a doubly transitive group of degree $5p + 6$. If H is primitive, it too may lead to a doubly transitive group of degree $5p + 6$. It is evident that the group of degree $5p + 6$, if it exists, is in neither case quadruply transitive nor is it a subgroup of a group of degree $5p + 7$. The possibility of the occurrence of the degree $5p + 6$ is due to the representation of the alternating and symmetric groups of degree 5 on six letters. It may be remarked that a substitution of order 3 and degree $3(p + 2)$ is present in this case.

STANFORD UNIVERSITY, CAL.

A PROOF OF THE INVARIANCE OF CERTAIN CONSTANTS OF ANALYSIS SITUS*

BY

J. W. ALEXANDER, II

1. **Introduction.** Every n-dimensional manifold may be partially characterized by means of two sets of topological invariants

$$P_1, P_2, \cdots P_{n-1} \qquad (P)\dagger$$

and

$$R_1, R_2, \cdots R_{n-1} \qquad (R)\ddagger$$

whose rôles are analogous to that played by connectivity in the theory of surfaces. The numbers P_i and R_i are each greater by unity than the maximum number of i-dimensional cycles (cf. § 4 below) which may be traced in the manifold, but in calculating the numbers P_i, certain conventions about sense are taken into account and the attention is confined to the sensed cycles.

Now if a manifold be subdivided into a complex, or generalized polyhedron, then one plus the maximum number of independent i-dimensional cycles of the polyhedron (i. e., cycles made up of cells of the polyhedron) is also equal to R_i, or to P_i if the conventions about sense be adopted. This theorem, which is of considerable use in calculating the values of the numbers (P) and (R), has been proved by Poincaré on the assumption that the manifold, all the cycles of the manifold, and all the cells of the complex may be regarded as made up of a finite number of analytic pieces. But such an assumption opens the way to a theoretical objection in that the numbers (P) and (R) when calculated from the analytic cycles alone might conceivably fail to be topological invariants. To remove this objection, it would have to be shown that there never could exist a point-for-point continuous reciprocal correspondence between two manifolds possessing different numbers (P) and (R), even if the correspondence were not required to be analytic.

In the following discussion, we shall take into account not only non-analytic cycles but also cycles possessing singularities of however complicated a nature.

* Presented to the Society, September 8, 1913.

† Poincaré, Journal de l'École Polytechnique, vol. 2 (1895), p. 19; and Palermo Rendiconti, vol. 13 (1899), p. 285.

‡ Veblen and Alexander, Annals of Mathematics, vol. 14 (1913), p. 168.

The inclusion of the latter makes it possible to prove with ease the fundamental theorem about the numbers (P) and (R) as well as a similar theorem about the *coefficients of torsion.** The discussion will very largely be limited to 3-dimensional manifolds, since the generalization affords no new difficulties and suggests itself at once.

The author wishes to express his appreciation and thanks to Professor O. Veblen for many valuable suggestions.

2. **Model of a 3-Dimensional Manifold.** From a topological point of view, a simple closed surface may always be represented in a space of sufficiently many dimensions by a model consisting of a finite number of triangular regions. These regions will abut upon one another in such a way that a side of one always coincides with a side of one of the others. Similarly, a 3-dimensional manifold may be represented by a finite number of tetrahedral regions, and so on.† As we are only concerned with manifolds without singularities, the regions which cluster about a common vertex may be supposed to constitute a simply connected portion of the manifold, that is, one which can be mapped upon the interior of a tetrahedron. We can then subdivide the manifold into smaller tetrahedral regions so as to assure ourselves that each region together with all the neighboring regions is contained within a simply connected portion. The model thus finally obtained will be used in place of the manifold itself in all of the discussion which follows.

The vertices, edges, and triangular faces which bound the tetrahedral regions subdivide the manifold into a *complex* π, or *generalized polyhedron*, of three dimensions. They constitute the *cells* of the polyhedron of dimensionalities 0, 1, and 2 respectively, while the interiors of the tetrahedral regions themselves constitute the cells of dimensionality 3. The polyhedron π is of a very restricted type; it can be assumed that all its edges are of equal length, for if its vertices are V in number, they may all be spaced at equal distances from one another in a space of $V - 1$ dimensions.

3. **Invariants of a Polyhedron and of a Manifold.** If a manifold be subdivided into a polyhedron κ, whether or not of the restricted type of π, it is a simple matter to calculate the greatest number of 1-dimensional complexes $c_1, c_2, \cdots c_k$ (closed curves) made up of cells of the polyhedron and such that there is no open 2-dimensional complex also made up of cells of the polyhedron and having for complete boundary one or more of the complexes $c_1, c_2, \cdots c_k$. Taking into account the conventions on sense, this calculation leads to an invariant $P_1^\kappa - 1$, otherwise to an invariant $R_1^\kappa - 1$. Similarly, the greatest number of independent closed 2-dimensional complexes is either $P_2^\kappa - 1$ or $R_2^\kappa - 1$. The problem will be to identify these four invari-

* Poincaré, Proceedings of the London Mathematical Society, vol. 32 (1900), p. 301.

† Veblen and Alexander, loc. cit., § 17.

ants with the corresponding invariants of the manifold itself, that is with $P_1 - 1$, $R_1 - 1$, $P_2 - 1$, and $R_2 - 1$.

4. By a 1- or 2-dimensional *cycle* c of the manifold will be meant a set of points of the manifold which may be regarded as the image of a closed complex λ of one or two dimensions as the case may be. There must be a continuous correspondence between the points of the complex and those of the cycle such that to every point of the complex there corresponds one and only one point of the cycle, but the correspondence need not be one-to-one in the inverse direction. Thus, the cycle may present singularities of any degree of complexity. Whenever two or more points of the complex λ correspond to the same point of the cycle c, we shall say that they correspond in a one-to-one manner to as many coincident points of c. With this convention, the correspondence between c and λ becomes one-to-one in both directions and the internal structure of c becomes the same as that of λ. If any length ϵ be preassigned, it may be assumed that the cells of the cycle c (the images of the cells of the complex λ) are so small that no two points of the same cell are at a distance of ϵ or more apart. For if we subdivide the cells of the complex λ into sufficiently small cells, the images of the latter will surely have the required property, owing to the uniform continuity of the correspondence between λ and c.

Before defining what is meant by an independent set of cycles, let us recall that the boundary of an open $i + 1$ dimensional complex κ consists of one or more closed i-dimensional complexes $\lambda_1, \lambda_2, \cdots \lambda_i$ any two of which may have in common one or more cells of dimensionality less than i. If such a complex be mapped along with its boundary upon the manifold, the complexes $\lambda_1, \lambda_2, \cdots \lambda_i$ will form a set of cycles $c_1, c_2, \cdots c_i$, two or more of which may coincide. When we disregard the conventions about sense, the cycle c_1 is said to be *dependent* upon the cycles $c_2, \cdots c_i$ provided it occurs an odd number of times in the set, but when the conventions are made, it is dependent provided the difference between the number of times it appears in one sense and the number of times it appears in the other is not zero. A set of cycles is said to be *independent* when no cycle of the set is dependent upon other cycles of the set. A necessary but not sufficient condition for independence is that each cycle of the set be non-bounding, that is that no cycle be the boundary of the image of a complex κ.

$R_1 - 1$ and $R_2 - 1$ are respectively the greatest number of independent 1- and 2-dimensions cycles which may be traced in the manifold; $P_1 - 1$ and $P_2 - 1$ are the corresponding numbers when conventions on sense are made.

5. **Lemma.** *No cycle lying in a simply connected region can be non-bounding.*

This may be seen at once if the simply connected region be regarded as the interior of a tetrahedron, for if we then join each point of the cycle by a linear

segment to some point P of the region, the points of the segments will constitute the image of an open complex bounded by the cycle in question. The complex will have as cells a vertex corresponding to the point P, an arc corresponding to each line joining P to an arc of the cycle, a triangular region corresponding to each set of lines joining P to an arc of the cycle, and, if the cycle be 2-dimensional, a tetrahedral region corresponding to each set of lines joining P to a triangular region of the cycle.

6. **The Fundamental Theorem for Polyhedra of the Restricted Type.** Let us first suppose that the manifold be subdivided into a polyhedron π of the type discussed in § 2. We shall first prove that every cycle of the manifold is dependent upon one or more cycles of π.

As was shown in § 4, the cells of the cycle c may be supposed to lie in regions as small as we please; hence, in particular, they may be supposed to lie in such small regions that if every point on the interior and boundary of any one of the cells be joined to the nearest vertex of the polyhedron, or to one of the nearest if there are more than one, these points will all be joined to the same or to adjacent vertices. Now if c be a one-dimensional cycle, the ends A_1 and A_2 of each one-cell will be joined respectively to two vertices B_1 and B_2 of the polyhedron π which either coincide or bound an edge $B_1 B_2$ of π. At all events, the circuit $A_1 A_2 B_2 B_1 A_1$ will bound the image s of some open complex since it lies in a simply connected region (§ 5). The totality of complexes s corresponding to the various circuits $A_1 A_2 B_2 B_1 A_1$ constitutes the image s of a complex bounded by the cycle c and, in general, one or more cycles of the polyhedron π composed of the arcs $B_1 B_2$. It may happen that the circuit c constitutes by itself the complete boundary of S, as for instance in the case where the vertices B_1 and B_2 all coincide. In any case, the cycle c is dependent upon the cycles of the polyhedron π.

When c is a 2-dimensional cycle, it may be regarded as composed of the images of triangular regions (§ 2). Each 2-cell will have three vertices A_1, A_2, and A_3 which will be joined to three vertices B_1, B_2, and B_3 of the polyhedron π every two of which must either coincide or bound an arc of π. Now, by the lemma, the cycles $A_1 A_2 B_2 B_1 A_1$, $A_2 A_3 B_3 B_2 A_2$, and $A_3 A_1 B_1 B_3 A_3$ bound three surfaces s_{12}, s_{23}, and s_{34} respectively all of which are situated within the same simply connected portion of the manifold. Moreover, the surfaces s_{12}, s_{23}, and s_{31}, together with the cells $A_1 A_2 A_3$ and $B_1 B_2 B_3$ (the latter being of less than two dimensions if two or more of the vertices B_1, B_2, and B_3 coincide) form a closed complex which bounds the image r of a 3-dimensional complex, again by the lemma. But the totality of regions r determines a region R which is bounded by the cycle c and, in general, one or more cycles c' of the polyhedron π made up of cells $B_1 B_2 B_3$.*

* The term *region* is here used in a very broad sense to denote the image, with or without singularities, of a 3-dimensional complex.

Since every cycle of the manifold is dependent upon one or more cycles of the polyhedron π, we have the relations

$$R_1^\pi \geqq R_1 \quad \text{and} \quad R_1^\pi \geqq R_1.$$

Moreover, we also have

$$P_2^\pi \geqq P_1 \quad \text{and} \quad P_2^\pi \geqq P_2,$$

since it is readily seen that the above argument holds equally well when the conventions about sense are introduced.

On the other hand, if a set of cycles a made up of cells of the polyhedron fails to be independent with respect to the manifold, so also does it fail to be independent with respect to the complex. For if it is not independent with respect to the manifold, there exists an open complex λ which may be so mapped upon the manifold that the boundary of its image c coincides with one or more of the cycles a. Let us first suppose that the cycles a are linear so that the complex λ is 2-dimensional. Then, supposing as we may that the cells are sufficiently small and the images of triangles, we may find a cell $B_1 B_2 B_3$ of the polyhedron π corresponding to every cell $A_1 A_2 A_3$ of c, by the process indicated above. The cells $B_1 B_2 B_3$, together with the cells of lower dimensionalities on their boundaries constitute one or more complexes c' of π bounded by cycles of the set a, therefore the cycles are not independent with respect to the polyhedron. It might be objected that the complex or complexes c' could fail to exist, as for instance if the vertices B_1, B_2, and B_3 always coincided. This can never be the case, however, for c and c' constitute the boundary of a certain open 3-dimensional region R, by the argument made above. But the boundary of R must be a closed complex, whereas c by itself is open. Therefore c' must not only exist but must have the same boundary as c in order that c and c' together shall constitute a closed complex.

When a is a set of 2-dimensional cycles, c is the image of a complex of three dimensions. We may therefore regard its cells as the images of tetrahedra, besides which we may suppose them as small as we please. If we join the vertices $A_1 A_2 A_3$ and A_4 of each cell to the four nearest vertices $B_1 B_2 B_3$ and B_4 of π respectively, the cells $B_1 B_2 B_3 B_4$ will determine the complex c'. The details of the argument in this and in the previous case are similar.

Thus, we also have

$$R_1^\pi \leqq R_1, \quad R_2^\pi \leqq R_2, \quad P_1^\pi \leqq P_1, \quad P_2^\pi \leqq P_2,$$

and therefore,

$$R_1^\pi = R_1, \quad R_2^\pi = R_2, \quad P_1^\pi = P_1, \quad P_2^\pi = P_2,$$

which establishes the theorem for the case where π is a polyhedron of the restricted type.

Evidently, all that has been said up to this point may be generalized at once to manifolds of more than three dimensions. The polyhedron π will then consist of simplexes, or generalized tetrahedra, together with the simplexes of lower dimensionalities on their boundaries.

7. **Corollary.** *If a manifold be topologically equivalent to the boundary of an n-dimensional simplex, or generalized tetrahedron, its invariants* (P) *and* (R) *are all equal to unity.*

For the cells on the boundary of the simplex determine a polyhedron of the restricted type π. Moreover, if we remove from one of the n-cells of the polyhedron a small n-dimensional simplex, there will be left a simply connected region containing all the cycles of the polyhedron. But, by the theorem, these cycles are the only ones which need be considered and, by the lemma, they all bound.

8. **The Fundamental Theorem for a General Polyhedron.** A polyhedron of the most general type may be transformed into one of the restricted type π by means of a series of regular subdivisions.* It merely remains to be shown, therefore, that a regular subdivision does not alter the number of independent cycles of the complex. Let κ' be the complex before the subdivision and let κ be the complex after the subdivision of a cell E of κ'. Then every i-dimensional cycle c_i of κ which was not a cycle of κ' previous to the subdivision must necessarily pass through the new vertex V of κ. Now the bases of the (pyramidal) cells of c_i which have for common apex the point V constitute one or more cycles d_{i-1} of dimensionality $i-1$, except for the case $i=1$ when they are a set of points, even in number. In either case, since the boundary of a cell is in point-for-point correspondence with the boundary of a simplex, the cycles or points d_{i-1} are the complete boundary of an open complex d_i made up of cells on the boundary of E (§ 7). But the complex d_i together with the cells of c_i which abut upon V constitutes a closed complex e_i which may be shown to bound by exactly the same argument as was used in proving the lemma. If we add† the complex e_i to the complex c_i, the latter is transformed into one which no longer passes through the vertex V and which therefore belonged to κ' previous to the subdivision.‡ Hence,

$$P_i^{\kappa} \leqq P_i^{\kappa'} \qquad \text{and} \qquad R_i^{\kappa} \leqq R_i^{\kappa'}.$$

Moreover, every cycle of κ' which bounds in κ also bounds in κ'. For either the bounded complex belonged to κ' or else it now passes through the

* A regular subdivision is made by partitioning one of the cells E of the complex into a set of "pyramidal" cells whose bases are the cells on the boundary of E and whose apexes coincide at a point V of E. For details, see Veblen and Alexander, loc. cit.

† Veblen and Alexander, loc. cit., p. 169.

‡ When the dimensionality of the cycle c_i is equal to or greater than that of the cell E, the cycle c_i includes all the sub-cells of E, therefore it is the same cycle as the cycle of κ' obtained by replacing the sub-cells of E by E itself.

vertex V. In the latter case, it can be replaced by one which does not pass through V by an argument like the one just made. Therefore,

$$P_i^\pi = P_i^{\pi'} \quad \text{and} \quad R_i^\pi = R_i^{\pi'}.$$

9. When the conventions on sense are introduced, it sometimes happens that a cycle is non-bounding when counted only once, though bounding when counted a sufficient number of times. From a consideration of such cycles are derived the coefficients of torsion of the manifold. An argument like the one made in this paper shows that a cycle having the above property may be replaced first by one of the same kind belonging to a polyhedron of type π, then by one of the same kind belonging to any polyhedron into which the manifold may be subdivided. The invariance of the coefficients of torsion therefore also follows at once.

PRINCETON, N. J.,
September, 1913.

POINTS SETS AND ALLIED CREMONA GROUPS*

(PART I)

BY

ARTHUR B. COBLE†

INTRODUCTION

The theory of invariants in the binary domain admits of generalization in two directions. If the emphasis be placed on the binary *form* of order n, the natural extension, and the one hitherto always considered, is to forms in a larger number of variables. If however the emphasis be laid on the set of n points, we are led to consider the invariant or projective properties of sets, P_n^k, of n points in S_k. Though it would seem that, since the P_n^k can be determined by a degenerate form of class k in S_k, the latter extension is contained in the former as a degenerate case, yet this is not true. The divergence of the two generalizations is due to the difference in their domains of rationality which in the first case is the domain of the coefficients of the form and in the second case is the domain of symmetric functions of the coördinates of the points of P_n^k. For example we shall find that P_6^2 has an invariant of degree two while the lowest degree of an invariant of the ternary sextic is three.

One object of this paper is to begin the formulation of a theory of invariants of the point set P_n^k. In § 3 and § 5 invariants of P_n^k are defined and complete systems for P_6^1 and P_6^2 are derived from this definition. The first two paragraphs are devoted to the *association* of a set P_n^k with a set Q_n^{n-k-2}. Sets which are thus associated have the same absolute invariants. Much of the material collected here would seem to be useful for other purposes. A study of the earlier cases indicates strongly that the lines followed in §§ 1, 2, and 9 will lead to important results in connection with the theta functions.

The first purpose, outlined above, is interrupted, perhaps rather rudely, in order to study an application of P_n^k to the theory of equations. The point set defines a Cremona group of order $n!$, $G_{n!}$, in a space $\Sigma_{k(n-k-2)}$. The $G_{n!}$ is a generalization of Moore's‡ "Cross-ratio Group." It is discussed

* Presented to the Society, April 25, 1914.

† This investigation is being pursued under the auspices of the Carnegie Institution of Washington, D. C.

‡ E. H. Moore, American Journal of Mathematics, vol. 22 (1900), p. 279.

in §§ 6, 7 while the form-problems of the group for sets P_5 and P_6 are considered in §§ 8, 10 in connection with related papers of the writer.*

If $n > k+3$, the $G_{n!}$ is a subgroup of a larger group in $\Sigma_{k(n-k-2)}$ which also is defined by P_n^k. This extended group is in general infinite and discontinuous though in certain very interesting cases it is finite. As n and k increase we obtain a doubly infinite series of extended groups which will be discussed in Part II of this paper.

For the particular set P_6^2 this extended group is finite and of order 51840. The rôle played by this group in the problem of the determination of the right lines on a cubic surface will be considered in Part III. For this purpose the matters developed in § 4 and § 5 of the present account will be of importance.

1. Associated and self-associated point sets

Let the set P_n^k of n points in S_k be given by the equations

$$(up_1) = 0, \quad (up_2) = 0, \quad \cdots, \quad (up_n) = 0.$$

These equations are connected by $n-k-1$ linear relations since in S_k any $k+2$ points must be linearly dependent. If we take these relations to be

$$q_{1i}(up_1) + q_{2i}(up_2) + \cdots + q_{ni}(up_n) \equiv 0 \qquad (i = 1, \cdots, n-k-1), \tag{1}$$

we can, by multiplying them respectively by v_i and adding, express them by the single identity in u and v

$$(vq_1)(up_1) + (vq_2)(up_2) + \cdots + (vq_n)(up_n) \equiv 0. \tag{2}$$

Thus the identities (1) lead to a set, Q_n^{n-k-2}, of n points in S_{n-k-2}. The identities may however be replaced by any $n-k-1$ independent linear combinations so that the points (vq) in S_{n-k-2} are determined only to within linear transformation. Moreover from the symmetry of (2) in points p and points q we can infer at once that the same projectively defined set Q_n^{n-k-2} is determined by any set projectively equivalent to P_n^k.

(3) *The class of projectively equivalent point sets P_n^k defines in* (2) *an "associated" class of projectively equivalent sets Q_n^{n-k-2} such that the relation between the two is mutual.*

Two point sets—one from each class—will be called *associated sets*. Unless the context clearly indicates the contrary it will be understood that a point set is any one of its class.

* *Cremona groups and the solution of equations*, vol. 9 (1908), p. 396; *The cross-ratio group and the solution of the sextic*, vol. 12 (1911), p. 311; *Finite geometry and theta functions*, vol. 14 (1913), p. 241; all in these Transactions. These papers are cited hereafter as *C*1, *C*2, *C*3 respectively.

If a P_4^1 is associated with a Q_4^1 the identity (2) reads

$$(4) \qquad (vq_1)(up_1) + \cdots + (vq_4)(up_4) \equiv 0.$$

Due to this identity all four pairs $(vq_i) \cdot (up_i)$ are apolar to the form $(ay)(bx)$ apolar to any three, i. e., the four pairs correspond in the binary projectivity, $(ay)(bx) = 0$, and the two sets are projective.

If a set P_n^1 is associated with Q_n^{n-3}, and if we utilize any $n-4$ of the points q, say $q_5, \cdots, q_n$, in order to form the contragredient symbol

$$(q_5, q_6, \cdots, q_n, y, y')$$

and then operate on (2) (y' operating on v), we obtain the new identity

$$(q_5, \cdots, q_n, y, q_1)(up_1) + (q_5, \cdots, q_n, y, q_2)(up_2) + \cdots + (q_5, \cdots, q_n, y, q_4)(up_4) \equiv 0.$$

Comparing this with (4) we see that the four S_{n-4}'s of the pencil on $q_5, \cdots, q_n$ which pass through $q_1, \cdots, q_4$ respectively are projective to $p_1, \cdots, p_4$ in S_1. Thus the set Q_n^{n-3} in S_{n-3} is projective on the rational norm-curve of order $n-3$ through it to the set P_n^1 in S_1. This property is characteristic since it requires that two sets Q_n^{n-3} and $Q_n'^{n-3}$ each associated with P_n^1 be projectively equivalent. Let us apply it to the case of the associated P_n^k and Q_n^{n-k-2}. In S_{n-k-2} let any $n-k-3$ of the points q be isolated, say $q_1, \cdots, q_{n-k-3}$. On these points there is a pencil of S_{n-k-3}'s, $k+3$ of which are determined by the points $q_{n-k-2}, \cdots, q_n$. By operating on (2) with $(q_1, \cdots, q_{n-k-3}, y, y')$ (y' operating on v), we obtain

$$(q_1, \cdots, q_{n-k-3}, y, q_{n-k-2})(up_{n-k-2}) + (q_1, \cdots, q_{n-k-3}, y, q_{n-k-1})(up_{n-k-1}) + \cdots + (q_1, \cdots, q_{n-k-3}, y, q_n)(up_n) \equiv 0.$$

Thus the $k+3$ members of the pencil (a $\bar{Q}_{k+3}^1$) are associated with the set $p_{n-k-2}, \cdots, p_n$ (a $\bar{P}_{k+3}^k$). Hence

(5) *If the sets P_n^k and Q_n^{n-k-2} are associated any group of $k+3$ points p are, when taken on the rational norm-curve through them, projective to the members of the pencil of S_{n-k-3}'s in S_{n-k-2} which are determined by the complementary set of $n-k-3$ points q and each in turn of the corresponding $k+3$ points q.*

Naturally the theorem (5) is still true if P_n^k and Q_n^{n-k-2} be interchanged and k be replaced by $n-k-2$. The property (5) is characteristic of associated sets. For if Q_n^{n-k-2} be given and $k+2$ points of P_n^k be fixed in S_k, the $n-k-2$ norm-curves on these $k+2$ points each carrying a further point of P_n^k are uniquely determined, and the position of the further point also is uniquely determined by the projective conditions on the parameters.

Another property of associated sets is the following:

(6) *If by projection from r of the points q, say $q_{n-r+1}, \cdots, q_n$, and section by an $S_{n-k-2-r}$ there is obtained from the $n-r$ remaining points q a set $Q_{n-r}^{n-k-2-r}$, this set is associated with the set P_{n-r}^k formed by the points $p_1, \cdots, p_{n-r}$ of P_n^k.*

This is proved by operating on the identity (2) with the symbol

$$(q_{n-r+1}, \cdots, q_n, y, y_1, \cdots, y_{n-k-2-r}),$$

y operating on v. In this symbol $y_1, \cdots, y_{n-k-2-r}$ can be regarded as a coördinate v' in $S_{n-k-2-r}$.

The identity expresses that the matrix M_p of n columns and $k+1$ rows formed from the coördinates of P_n^k and the matrix M_q of n columns and $n-k-1$ rows formed from the coördinates of Q_n^{n-k-2} are conjugate. The theorem of Grassman that

(7) *Complementary determinants of the matrices M_p and M_q are proportional* can be proved at once by means of the identity. For if we operate on (2) with the symbol $(q_1, q_2, \cdots, q_{n-k-2}, y)(x, p_{n-k+1}, \cdots, p_n)$, the y operating on v and the x on u, we get

$$(q_1, \cdots, q_{n-k-2}, q_{n-k-1})(p_{n-k-1}, p_{n-k+1}, \cdots, p_n) + (q_1, \cdots, q_{n-k-2}, q_{n-k})(p_{n-k}, p_{n-k+1}, \cdots, p_n) = 0.$$

If we set

$$(q_1, q_2, \cdots, q_{n-k-2}, q_{n-k-1}) = \lambda(p_{n-k}, p_{n-k+1}, \cdots, p_n),$$

then

$$(q_1, q_2, \cdots, q_{n-k-2}, q_{n-k}) = -\lambda(p_{n-k-1}, p_{n-k+1}, \cdots, p_n).$$

Other equalities can be deduced from these as the second is deduced from the first until the theorem is completely established.

Let us define the number

$$N_{n,k} = 2k + 2 - n \tag{8}$$

to be the *norm* of P_n^k. For associated sets,

$$N_{n,k} + N_{n,n-k-2} = 0. \tag{9}$$

In many respects those sets whose norm is zero have the simplest projective properties. Given then the set P_{2k+2}^k let us obtain a construction for its associated Q_{2k+2}^k. Suppose that the two sets are in the same S_k so that the first $k+2$ points of the one set coincide respectively with the corresponding first $k+2$ points of the other and form in S_k a *base* (i. e., $k+2$ points such that no $k+1$ lie in an S_{k-1}). A manifold containing the points of the base will be called a *basic* manifold. We shall need the following theorems:*

(*a*) *Basic rational norm-curves, R^k, which touch an S_{k-1}, α, touch at points of a quadric, Q_α, in α.*

* For these three theorems and further references, see Conner, *Basic systems of rational norm-curves*, American Journal of Mathematics, vol. 32 (1911), p. 167.

(*b*) *Basic quadrics meet* α *in quadrics apolar to* Q_α.

(*c*) *Basic* R^k*'s meet* α *in point sets* P_k^{k-1} *self polar as to* Q_α.

The quadric Q_α is defined as a locus of points by (*a*) and as an envelope of S_{k-2}'s by (*b*).

We shall need also the

LEMMA. *If the members of a pencil of* S_{k-1}*'s on an* S_{k-2} *and the* $k+2$ *base points are projective to the parameters of the base points on a basic* R^k, *the* S_{k-2} *cuts* R^k *in* $k-1$ *points.*

For if y is the parameter of the pencil, x the parameter on R^k and $y = x$ the projectivity, then the sets cut out on R^k by members of the pencil are determined by $(ay)(bx)^k = 0$. Since this must vanish $k+2$ times for $y = x$ we must have $(ax)(bx)^k \equiv 0$ whence $(ay)(bx)^k = (yx) \cdot (cx)^{k-1}$. Hence every S_{k-1} and therefore their common S_{k-2} cuts R^k in the $k-1$ points $(cx)^{k-1} = 0$.

According to (5) the pencil of S_{k-1}'s on the S_{k-2} containing $q_{k+4}, \cdots, q_{2k+2}$ and also in turn the points $q_1 = p_1, \cdots, q_{k+2} = p_{k+2}, q_{k+3}$ is projective to the set of parameters of $p_1, \cdots, p_{k+2}, p_{k+3}$ on the R^k through them. According to the lemma S_{k-2} must cut R^k in $k-1$ points and the particular $S_{k-1}(\alpha)$ which is on q_{k+3} must then contain p_{k+3}. But α is determined by $q_{k+3}, \cdots, q_{2k+2}$ and therefore contains all the points $p_{k+3}, \cdots, p_{2k+2}$. Moreover according to (*c*) the S_{k-2} in α which contains $q_{k+4}, \cdots, q_{2k+2}$ is the polar S_{k-2} of p_{k+3} as to Q_α. Hence

(10) *If the associated sets* P_{2k+2}^k *and* Q_{2k+2}^k *be placed so that the first* $k+2$ *points of each constitute the same base in* S_k *the remaining* k *points of each set lie in the same* S_{k-1}, α, *and form polar* k*-points of the quadric* Q_α *in* α *which is apolar to all the sections by* α *of basic quadrics.*

An inversion of the argument shows that this property implies the property of association. The construction in (10) can be used in connection with (6) to construct the associated sets P_n^k and Q_n^{k-k-2} whose norm is not zero. Suppose that the norm N of P_n^k is positive which according to (9) requires only a proper naming of the sets. Since $2k+2 = n+N$, P_n^k can be gotten by dropping say the last N points of P_{2k+2}^k in (10). If then Q_{2k+2}^k as constructed in (10) be projected from its last N points upon the S_{n-k-2} determined by the first $n-k-1$ base points we obtain a Q_n^{n-k-2} which is associated with P_n^k according to (6) and whose first $n-k-1$ points coincide with the corresponding points of P_n^k.

The striking feature of the P_{2k+2}^k with zero norm is that it may entirely coincide, i. e., be projective in the identical order, with its associated set. If this occurs P_{2k+2}^k will be called a *self-associated set.** Then in the identity

* The much discussed P_8^3, the base points of a net of quadrics, is usually called an "associated set."

(2) we can take (vq_i) to be $\lambda_i(up_i)$ and the identity becomes

$$\lambda_1(up_1)^2 + \lambda_2(up_2)^2 + \cdots + \lambda_{2k+2}(up_{2k+2})^2 \equiv 0. \tag{11}$$

From this identity and from (10) we have

(12) *The self-associated set P^k_{2k+2} is defined by the property that all the quadrics on any $2k+1$ of the points pass through the remaining point. It is also defined by the property that the S_{k-1}, α, on any k of the points is cut by quadrics on the remaining $k+2$ points in sections all apolar to a definite quadric in α, Q_a, of which the k points form a self-polar k-edron. It depends upon $\frac{1}{2}k(k+1)$ absolute constants.**

After fixing the $k+2$ base points the number of absolute constants appears as the number k involved in the choice of α which also determines Q_a, and the number $\frac{1}{2}k(k-1)$ involved in the choice of a self polar k-edron of Q_a.

The P^1_4, any 4 points on a line, is a very special case which appears under (11). The P^2_6 on a conic is the next case. More generally any $2k+2$ points on the rational norm-curve R^k in S_k constitute a self-associated set [Cf. (*c*) above]. Again the $2k+2$ points cut out on an elliptic norm-curve, E^{k+1}, in S_k by a quadric are self-associated. This set is known to be general in S_2 and S_3 and can be proved to be the general set in S_4, but for higher dimensions it must be a special self-associated set since it contains only $2k+2$ absolute constants.

The association of point sets implies a mutual ordering of the sets. Self-association implies that the associated sets are also projective in the given order which will be referred to as the identical order. The associated sets may be projective in some other order and then the one set can be projected upon the other. Such a set will be said to be *self-associated in other than the identical order.* Let us consider the possible cases for the P^1_4 and the P^2_6. First we may note generally that

(13) *If a set P^k_{2k+2} on an R^k is self-associated in an order other than the identical one it is also self-projective in this order and conversely.*

For we have already noticed that such a set is self-associated in the identical order and two point sets each associated with a third are projective. Thus the identity (2) furnishes a convenient algebraic condition for the self-projectivity of such a set. From the known properties of the binary quartic we have

(14) *The general set P^1_4 is self-associated in four distinct orders; the harmonic P^1_4 in eight distinct orders; and the equianharmonic P^1_4 in twelve distinct orders.*

If the P^2_6 is on a conic, (13) leads to the self projective types of Bolza. Let us require then of P^2_6 that no two points coincide, that no three be on a line, and that the six be not on a conic. The identity (2) can be replaced by the

* For $k=3$, cf. Serret, *Géométrie de direction*, pp. 313–16.

two identities

$$\lambda_1(up_1)(uq_1) + \cdots + \lambda_6(up_6)(uq_6) \equiv 0,$$
$$\lambda_1(p_1 q_1 x) + \cdots + \lambda_6(p_6 q_6 x) \equiv 0, \tag{15}$$

where the points q are some permutation of the points p. Hereafter the points p are indicated merely by their subscripts. The possible orders in which Q_6^2 can coincide with P_6^2 are (12)(34)(56), (12)(34), (12), (123)(45), (123)(456), (123), (1234)(56), (1234), (12345), and (123456). Here, e. g., (123) indicates that 1, 2, 3, 4, 5, 6 are associated with 2, 3, 1, 4, 5, 6.

The order (12)(34)(56) entails the identities:

$$(\lambda_1 + \lambda_2)(u1)(u2) + (\lambda_3 + \lambda_4)(u3)(u4) + (\lambda_5 + \lambda_6)(u5)(u6) \equiv 0,$$
$$(\lambda_1 - \lambda_2)(12x) + (\lambda_3 - \lambda_4)(34x) + (\lambda_5 - \lambda_6)(56x) \equiv 0.$$

If all of the differences vanish the first identity requires that P_6^2 be the vertices of a four line. If only the first two differences vanish, 5 and 6 coincide. If only the first difference vanishes either two points coincide or 3, 4, 5, 6 are on a line. All of these cases have been excluded. If none of the differences vanish, the lines $\overline{12}$, $\overline{34}$, $\overline{56}$, meet in a point and both identities can be satisfied by the choice, $\lambda_2 = -\lambda_1, \lambda_4 = -\lambda_3, \lambda_6 = -\lambda_5$.

The order (12)(34) entails the identities:

$$(\lambda_1 + \lambda_2)(u1)(u2) + (\lambda_3 + \lambda_4)(u3)(u4) + \lambda_5(u5)^2 + \lambda_6(u6)^2 \equiv 0,$$
$$(\lambda_1 - \lambda_2)(12x) + (\lambda_3 - \lambda_4)(34x) \equiv 0.$$

Unless 1, 2, 3, 4 are on a line or unless two points coincide we must take $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. According to the first identity the pairs 1, 2 and 3, 4 are apolar to a net of conics on 5 and 6. They must be either pairs on the conic C through 5, 6 which with the line 56 makes up the jacobian of the net, or pairs on the line 56, or a pair on C and a pair on 56, all of which cases are excluded.

The order (12) if 1, 2 do not coincide implies the identity:

$$2\lambda_1(u1)(u2) + \lambda_3(u3)^2 + \cdots + \lambda_6(u6)^2 \equiv 0.$$

Thus 1, 2 are apolar to the pencil of conics on 3, 4, 5, 6; i. e., they correspond in the quadratic Cremona involution with the remaining four fixed points.

For the order (123) the second identity reads as follows:

$$\lambda_1(12x) + \lambda_2(23x) + \lambda_3(31x) \equiv 0,$$

which requires that 1, 2, 3 be on a line.

The first identity for the order (123)(456) shows that the two triangles

are self polar as to a conic which is necessarily proper unless two points coincide or three are on a line. Hence the six points are on a conic.

In the case of the order (123)(45) we can use the fact [cf. *C*1, p. 398] that a Cremona transformation T of the fifth order whose double singular points are P_6^2 can be constructed so that the double singular points of T^{-1} are Q_6^2. Then T^2 is a collineation which effects upon the points the square of the permutation (123)(45). If 1, 2, 3 are not on a line this collineation of period three has 4, 5, 6 for fixed triangle and P_6^2 is a pair of flex triangles of a cubic. We shall see below that this condition is sufficient.

A similar argument for the order (1234)(56) shows that a harmonic perspectivity with center $C \equiv \overline{\overline{13}\,\overline{24}}$ and axis $\overline{56}$ must interchange 1, 3 and also 2, 4. According to the first identity either all the conics apolar to the pairs 12, 23, 34, 41 must be apolar to the pair 56 or, if $\lambda_5 + \lambda_6 = 0$, all the conics apolar to 12, 23, 34 must be apolar to 41. The first case is impossible since then the lines $\overline{13}$ and $\overline{42}$ each have two poles as to such conics and the conics must be the line pairs on C harmonic with $\overline{13}$ and $\overline{42}$ which are not all apolar to 56. The second case also is impossible since there is in general only a pencil of conics apolar to the four pairs.

The case (12345) also is not possible if the P_6^2 is not on a conic. For T^2 is a cyclic collineation which is not an homology, say $x_1' = x_1$, $x_2' = \epsilon x_2$, $x_3' = \epsilon^4 x_3$ ($\epsilon = e^{2\pi i/5}$). Then the points $1, \cdots, 5$ are $1, \epsilon^\nu, \epsilon^{4\nu}$ ($\nu = 0, \cdots, 4$) while 6 must be 1, 0, 0. The parameters of $1, \cdots, 5$ on their conic are $t = 1, \epsilon^4, \epsilon^3, \epsilon^2, \epsilon$. The parameters of the pencil from 6 to $1, \cdots, 5$ are $t = 1, \epsilon^2, \epsilon^4, \epsilon, \epsilon^3$. These two sets in the proper order are not projective.

Taking up the case (1234) we find from T^2 that the pairs 1, 3 and 2, 4 correspond in a harmonic perspectivity with axis $\overline{56}$. If we take the points to be

$$1, 1, 1; \quad y_1, y_2, y_3; \quad -1, 1, 1; \quad -y_1, y_2, y_3; \quad 0, 1, 0; \quad 0, 0, 1,$$

we find from (2) that necessarily $y_2 + y_3 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, $\lambda_5 = -4, \lambda_6 = 4$. Thus 5, 6 must harmonically separate the lines $\overline{13}$ and $\overline{42}$ which implies that P_6^2 is self-associated in the orders (1536) and (4526) as well. If it be required in addition that P_6^2 be self-associated in the order (2463) we find that $y_1 : y_2 : y_3 = -2 \pm \sqrt{5} : 1 : -1$. Then P_6^2 is the Clebsch six-point. If we take it in the canonical form

$$(up_\infty) = u_1, \qquad (up_i) = u_1 + \epsilon^{4i} u_2 + \epsilon^i u_3 \quad (i = 0, \cdots, 4),$$
$$\epsilon = e^{2\pi i/5},$$

the identity (2) for the permutation (1342) reads:

$$-5(u\infty)(v\infty) + (u0)(v0) + (u1)(v3) + (u2)(v1) + (u3)(v4) + (u4)(v2) \equiv 0.$$

Since this P_6^2 is self-projective in 60 ways we find by using the projectivities two more distinct ways of self-association, namely $(\infty 0)(34)(12)$ and $(\infty 23140)$.

Let us consider finally the order (123456). Now T^2 is a collineation of period three and T^3 is a Cremona transformation for which the points are associated in the order $(14)(25)(36)$ which requires that the lines $\overline{14}$, $\overline{25}$, $\overline{36}$ meet in a point. Thus the P_6^2 can be thrown into the form

$$1, 1, 1; \quad 1, \omega, \omega^2; \quad 1, \omega^2, \omega; \quad \rho, 1, 1; \quad \rho, \omega, \omega^2; \quad \rho, \omega^2, \omega;$$

$$\omega = e^{2\pi i/3}.$$

One verifies easily that the identity (2) holds for the values

$$\lambda_1 = \lambda_3 = \lambda_5 = -\lambda_2 = -\lambda_4 = -\lambda_6.$$

If $\rho = \omega$, we have two flex triangles. These are self-projective in 36 ways. By combining the projectivities with the present case of self-association we find that this P_6^2 is self-associated in the further types of order, (26), $(135)(26)$, and $(14)(36)(25)$. Hence

(16) *A P_6^2 for which no two points coincide, no three are on a line, and the six are not on a conic, can be associated with itself only in the following odd orders; (a) in $(12)(34)(56)$ if the lines $\overline{12}$, $\overline{34}$, $\overline{56}$ meet in a point (3 absolute constants); (b) in (56) if 5 and 6 correspond in the quadratic involution with fixed points at 1, 2, 3, 4 (2 absolute constants); (c) in (1234) if 1, 3 and 4, 2 are corresponding pairs in the reflexion with axis 56 and center $\overline{\overline{13}, \overline{42}}$ and if the pair 5, 6 is apolar to the conic $\overline{13}, \overline{42}$ (1 absolute constant); (d) in (123456) if 1, 3, 5 and 2, 4, 6 are cyclically permuted by a collineation of period three with a fixed point on the lines $\overline{14}$, $\overline{25}$, $\overline{36}$ (1 absolute constant). Furthermore two flex triangles 1, 3, 5 and 2, 4, 6 are self-associated in the orders, (123456), (26), $(135)(26)$, and $(14)(36)(25)$; and a Clebsch six-point, ∞, 0, 1, 2, 3, 4, is self-associated in the orders, (1342), $(\infty 0)(34)(12)$, and $(\infty 23140)$*

*If however P_6^2 is on a conic it is self-associated in the identical order and in any other order according to which the six points or their parameters are self-projective. These types are furnished by Bolza's types of self-projective binary sextics.**

2. Irrational conditions for associated and self-associated point sets

Let P_n^k and Q_n^{n-k-2} be two associated sets for which, to fix ideas, the norm N of P_n^k is not greater than zero. If from the S_{k-j}, $u = c_0, c_1, \cdots, c_{k-j}$, determined by any $k - j + 1$ points of P_n^k numbered $c_0, \cdots, c_{k-j}$ we project

* A type of self-conjugate association is treated by E. Study, *Kürzeste Wege im komplexen Gebiet*, Mathematische Annalen, vol. 60 (1905), p. 321. For references to earlier articles on association see his footnote p. 348.

$2j$ points of P_n^k, say those numbered $a_1, \cdots, a_j, b_1, \cdots, b_j$, upon an S_{j-1} we obtain $2j$ points which are connected by the $\binom{2j}{j-1}$ determinant identities which pertain to S_{j-1}. If we denote by $P(i, \cdots, l)$ the $(k+1)$-rowed determinant formed from the coördinates of $(up_i), \cdots, (up_l)$, the corresponding identities in S_k have the form

$$(17)\quad \begin{aligned} &P(a_1, \cdots, a_j, u)\,P(b_1, \cdots, b_j, u) \\ &\quad - P(b_1, a_1, \cdots, a_{j-1}, u)\,P(a_j, b_2, \cdots, b_j, u) + \cdots \\ &\quad + (-1)^j P(a_2, \cdots, a_j, b_1, u)\,P(a_1, b_2, \cdots, b_j, u) = 0, \end{aligned}$$

the terms being obtained from the first by the cyclic advance of $a_1, \cdots, a_j, b_1$. If the remaining points of P_n^k are numbered $d_0, \cdots, d_{n-k-2-j}$ and if

$$v = \overline{d_0, \cdots, d_{n-k-2-j}}$$

is the $S_{n-k-2-j}$ in S_{n-k-2} which contains the corresponding points of Q_n^{n-k-2}, then according to the statement (7) we can replace the determinants above such as $P(a_1, \cdots, a_j, u)$ by determinants such as $Q(b_1, \cdots, b_j, v)$ if account be taken of the factor of proportionality (which may be neglected in any homogeneous relation) and the sign. If then each product in (17) be multiplied by the complementary product from Q_n^{n-k-2} and if the square root be taken, the identity is unaltered for proper choice of the signs of the radical and radicand. Then (17) takes the form

$$(18)\quad \begin{aligned} &\sum \sqrt{P(a_1, \cdots, a_j, u)\,Q(b_1, \cdots, b_j, v)}\,\cdot \\ &\sqrt{P(b_1, \cdots, b_j, u)\,Q(a_1, \cdots, a_j, v)} = 0, \end{aligned}$$

where $\sum$ refers to the cyclic advance of $a_1, \cdots, a_j, b_1$. We observe that if any change of sign occurs in the substitution of the complementary determinants as we pass from one term to the next this change must occur twice in the same product.

(19) *The coördinates of the associated sets P_n^k and Q_n^{n-k-2} satisfy the*

$$\binom{n}{k-j+1}\binom{n-(k-j+1)}{n-k-1-j}\binom{2j}{j-1}$$

irrational relations of type (18) *for $j = 2, \cdots, k+1$ if $N_{n,k} < 0$. A particular relation is determined by the choice of the $k-j+1$ points of P_n^k on u, of the $n-k-1-j$ complementary points of Q_n^{n-k-2} on v, and of the $j-1$ points $b_2, \cdots, b_j$ from the remaining $2j$ points of the sets. For any value of j these relations are sufficient to establish the association.*

Let us prove that the relations for $j = 2$ are sufficient. They require that if $p_1, \cdots, p_4$ be projected from $k-1$ points of P_n on u into points $\alpha_1, \cdots, \alpha_4$ of a line, and if $q_1, \cdots, q_4$ be projected from the complementary

$n-k-3$ points of Q_n^{n-k-2} on v into points $\beta_1, \cdots, \beta_4$ of a line, then

$$\sqrt{(\alpha_2\,\alpha_3)(\alpha_1\,\alpha_4)\cdot(\beta_2\,\beta_3)(\beta_1\,\beta_4)}+\sqrt{(\alpha_3\,\alpha_1)(\alpha_2\,\alpha_4)\cdot(\beta_3\,\beta_1)(\beta_2\,\beta_4)}$$
$$+\sqrt{(\alpha_1\,\alpha_2)(\alpha_3\,\alpha_4)\cdot(\beta_1\,\beta_2)(\beta_3\,\beta_4)}=0.$$

This can be written $\sqrt{x_1\,y_1}+\sqrt{x_2\,y_2}+\sqrt{x_3\,y_3}=0$ where $x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0$. Rationalizing and eliminating x_1 and y_1 we find that $x_2/x_3=y_2/y_3$, i. e.,

$$\frac{(\alpha_3\,\alpha_1)(\alpha_2\,\alpha_4)}{(\alpha_1\,\alpha_2)(\alpha_3\,\alpha_4)}=\frac{(\beta_3\,\beta_1)(\beta_2\,\beta_4)}{(\beta_1\,\beta_2)(\beta_3\,\beta_4)}.$$

Hence the points α and the points β are projective. This being true of any four points and any choice of the points of projection we verify at once that the characteristic property (5) of associated sets holds for the given case. The value $j=k+1$ for $N_{n,\,k}=0$ is exceptional. In that case the relations (18) are satisfied when the associated sets P_{k2+2}^k and Q_{2k+2}^k are projective or when they are associated.

If for $N=0$ the set P_{2k+2}^k is a self-associated set then it can be brought into coincidence with its associated set Q_{2k+2}^k whence

(20) *The coördinates of the self-associated set P_{2k+2}^k satisfy the*

$$\tfrac{1}{2}\binom{2k+2}{k-j+1}\binom{k+j+1}{k-j+1}\binom{2j}{j-1} \qquad (j=2,\cdots,k)$$

irrational relations obtained from (18) *for $Q=P$. These conditions for a given value of j are sufficient for self-association.*

So far as the writer is aware these irrational conditions for association and self-association are novel. However for the P_6^2 on a conic one is sufficient and a condition closely related to that given above appears in connection with Kummer's surface.*

3. Invariants of point sets. The complete system of P_6^1

The definition of the rational integral invariants of the set P_n^1, i. e., of the binary form of order n, in terms of the particular points p can be generalized as follows:

(21) *A rational integral invariant of the point set P_n^k is a rational integral function of the $\binom{n}{k+1}$ determinants $P(i_1, i_2, \cdots, i_{k+1})$ which is (A) homogeneous and of the same degree in the coördinates of each of the n points and (B) unaltered in value however the points be permuted.*

Thus the general form of an invariant of the set is

$$(22) \qquad I=\sum\pi P(i_1, i_2, \cdots, i_{k+1})^{a_{i_1 i_2 \cdots i_{k+1}}},$$

* Cf. Hudson, *Kummer's Quartic Surface*, pp. 135–6 and elsewhere.

where according to (A) in each product π of $\sum$ the $\binom{n}{k+1}$ positive integers $a_{i_1 i_2 \cdots i_{k+1}}$ satisfy the diophantine system of n equations

$$\sum_j a_{i_1 i_2 \cdots i_{k+1}} = \nu \qquad (j = 1, \cdots, n). \tag{23}$$

Here $\sum_j$ indicates the sum of all the exponents a which have a subscript equal to j and ν is the *degree of the invariant*, i. e., its degree in the coördinates of a particular point. According to (B), $\sum$ in (22) refers to any sum of such products π, which is invariant under permutations of the points. Obviously under linear transformation I is unaltered to within a power of the determinant of the transformation. The index of this power is the *weight*, w, of I. Then

$$w = \frac{n\nu}{k+1}. \tag{24}$$

An *irrational integral invariant* of P_n^k is a function which satisfies all the requirements of (21) except (B). An *incomplete integral invariant* (rational or irrational as the case may be) of P_n^k is one which satisfies the proper requirements for a set $P_{n'}^k (n' < n)$ contained in P_n^k. *Fractional* and *absolute* invariants are defined as is customary. Under the above definitions a cross-ratio of four points of a P_n^1 $(n > 4)$ would be described as an incomplete irrational absolute invariant of P_n^1.

From every invariant of any type of P_n^k there is obtained by passing to the complementary determinants an invariant J of the same type of the associated set Q_n^{n-k-2} and conversely. Since the weights are the same the degrees are connected by the relation $\nu_J/\nu_I = n/(k+1) - 1$.

(25) *The corresponding invariants I and J of the associated sets P_n^k, Q_n^{n-k-2} are equal to within a factor which is a power (whose index is the weight) of an undetermined constant. To a syzygy among the invariants I there corresponds a syzygy among the invariants J; to a complete system of invariants I, a complete system of invariants J.*

As an example of the determination of a complete system of invariants of an n-point, and as a means of introducing some formulæ for use in the next paragraph, let us take the case of a P_6^1 with coördinates p_{i1}, p_{i2} $(i = 1, \cdots, 6)$. The general invariant has the form

$$I = \sum \pi (ik)^{a_{ik}}, \qquad (ik) = (p_{i1} p_{k2} - p_{k1} p_{i2}),$$

where π satisfies the diophantine system

$$j^\circ. \quad a_{j1} + a_{j2} + \cdots + a_{j6} = \nu, \quad a_{jj} = 0, \quad a_{ij} = a_{ji} \qquad (j = 1, \cdots, 6).$$

Let us first prove that

7° *The product π can be expressed rationally and integrally in terms of the fifteen products of the type $(ij)(kl)(mn)$.* We shall say that π is "reducible" according to these fifteen "moduli."

(*a*) If $a_{12} = \nu$, then $a_{1i} = 0$, $a_{2i} = 0$ $(i = 3, \cdots, 6)$. From $4° + 5° + 6° - 3°$ we find that $a_{45} + a_{56} + a_{64} = \nu$.

(a_1) If $a_{45} = \nu$, then $a_{5i} = 0$, $a_{5i} = 0$ and from $6°$, $a_{36} = \nu$ whence π is reducible mod. (12)(45)(36).

(a_2) If $0 < a_{45} < \nu$ and $0 < a_{56} < \nu$ then $a_{36} = a_{34} = 0$ else π is reducible mod. (12)(45)(36) or mod. (12)(56)(34). Then $a_{35} = \nu$ which contradicts $0 < a_{45} < \nu$, due to $5°$.

Thus π is reducible if any $a_{ik} = \nu$.

(*b*) Let $0 < a_{12} < \nu$. From $3° + 4° + 5° + 6° - 1° - 2°$ we have

$$a_{34} + a_{35} + a_{36} + a_{45} + a_{56} + a_{64} = \nu + a_{12}.$$

Hence at least two a's on the left are greater than zero. Let $0 < a_{34} < \nu$ whence $a_{56} = 0$ else π is reducible mod. (12)(34)(56). Let $0 < a_{35} < \nu$ whence $a_{46} = 0$ else π is reducible. From $1° + 2° + 3° - 4° - 5° - 6°$ we now get $a_{12} + a_{13} + a_{23} - a_{45} = 0$, whence $0 < a_{45} < \nu$ and further $a_{36} = 0$ if π is not reducible. Now $6°$ reads as follows: $a_{61} + a_{62} = \nu$ and therefore $0 < a_{61} < \nu$ and $0 < a_{62} < \nu$. Thus if π is not reducible,

$$a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = 0.$$

Then the solution of the system is

$$a_{12} = a_{16} = a_{26} = a_{34} = a_{35} = a_{45} = \nu/2.$$

Now π contains the factor

$$(12)(16)(26)(34)(35)(45) = (16)(26)(35)(45)[(13)(24) - (14)(23)]$$

and is reducible. The coefficients of the moduli thus introduced in π must satisfy a similar diophantine system for a smaller value of ν and the process can be repeated until the expression required in $7°$ is obtained. Hence

(26) *Any invariant I of* P_6^1 *is a rational integral function of the fifteen products* $(ij)(kl)(mn)$, *which is symmetric in the points.*

If now we form from these products the six functions of Joubert,*

$$\begin{aligned}
A &= (25)(13)(46) + (51)(42)(36) + (14)(35)(26) + (43)(21)(56) + (32)(54)(16),\\
B &= (53)(12)(46) + (14)(23)(56) + (25)(34)(16) + (31)(45)(26) + (42)(51)(36),\\
C &= (53)(41)(26) + (34)(25)(16) + (42)(13)(56) + (21)(54)(36) + (15)(32)(46),\\
D &= (45)(31)(26) + (53)(24)(16) + (41)(25)(36) + (32)(15)(46) + (21)(43)(56),\\
E &= (31)(24)(56) + (12)(53)(46) + (25)(41)(36) + (54)(32)(16) + (43)(15)(26),\\
F &= (42)(35)(16) + (23)(14)(56) + (31)(52)(46) + (15)(43)(26) + (54)(21)(36),
\end{aligned} \tag{27}$$

* These functions have been used by Joubert and Richmond (for references see *C*2) to obtain invariants of the sextic but not to prove the completeness of the system.

we verify at once that an even permutation of the points effects an even permutation of the functions and that an odd permutation of the points effects an odd permutation of the functions accompanied by a change of sign throughout. A set of parallel generating permutations is

$$(28) \qquad (12),\ (23456) : (AD)(BE)(CF),\ (ADBFE).$$

The six points and functions define the two sets of six conjugate ikosahedral subgroups in the even permutation group, g_{360}. The fundamental binary identity leads at once to the relation $A + D = 4(43)(21)(56)$ and by applying (28) we get the conjugate set,

$$\begin{aligned}
&A + B = 4(51)(43)(26), && B + C = 4(34)(25)(16),\\
&A + C = 4(53)(41)(26), && B + D = 4(31)(45)(26),\\
&A + D = 4(43)(21)(56), && B + E = 4(12)(53)(46),\\
&A + E = 4(32)(54)(16), && B + F = 4(23)(14)(56),\\
&A + F = 4(25)(13)(46), && C + D = 4(15)(32)(46),\\
&C + E = 4(13)(42)(56),\\
&C + F = 4(54)(21)(36),\\
&D + E = 4(25)(41)(36),\\
&D + F = 4(24)(53)(16),\\
&E + F = 4(15)(43)(26).
\end{aligned} \qquad (29)$$

Thus the fifteen products can be expressed rationally and integrally in terms of $A, \cdots, F$. Then the half symmetric invariants are rational and integral in $\sum A, \sum A^2, \sum A^3, \sum A^4, \sum A^5, \sum A^6$ and $\pi(A - B)$. Of these $\sum A, \sum A^3, \sum A^5$ change sign under odd permutation of the points and therefore must contain $\sqrt{D} = \pi(ik)$ as a factor. The weight of the first two is too low to permit this whence $\sum A = \sum A^3 = 0$ while $\sum A^5 = \lambda\sqrt{D}$. This shows that

(30) *The invariants* $\sum A^2, \sum A^4, \sum A^6, (\sum A^5)^2$, *and* $\pi(A - B)$ *constitute a complete system of rational integral invariants of* P_6^1, *i. e. of the binary sextic. The square of the last is rationally expressible in terms of the others.*

By using (29) we find that the factor $A - B$ of the skew invariant $\pi(A - B)$ has the value

$$A - B = (A + C) - (B + C) = \cdots = 4[(53)(41)(26) - (34)(25)(16)] = \cdots.$$

If it vanishes the pairs 15, 24, 36 are in an involution.

By means of this method it is quite easy to derive and to prove the completeness of the system of invariants and covariants of the binary cubic and quartic.

4. Lines and tritangent planes of a cubic surface mapped from a plane by cubic curves on six points

The principle of transference of Clebsch applied to the functions of Joubert leads to the result:

(31) *If in the functions, $A, \cdots, F$, each binary determinant (ik) be replaced by the ternary determinant (ikx), there arise six cubic curves, $a = 0, \cdots, f = 0$, on six points, $p_1, \cdots, p_6$ of a plane π, which are connected by the identities* $\sum a = 0$, $\sum a^3 = 0$, $\sum a^5 = \lambda\pi(ikx)$, $a + b = 4(51x)(42x)(36x)$, $\cdots$.

Six cubic curves on six points are connected by two linear relations. One of these is $\sum a = 0$. If we take the other to be $\sum \bar{a}a = 0$, the coefficients $\bar{a}, \cdots, \bar{f}$ can be fixed to within a factor of proportionality by the requirement that $\sum \bar{a} = 0$.

Let us denote by (ij, kl, mn) the ternary determinant whose vanishing expresses that the lines (ijx), (klx), (mnx) meet in a point. Then the fifteen relations

$$
\begin{aligned}
&\bar{a} - \bar{b} = (15, 24, 36), \qquad \bar{b} - \bar{c} = (25, 34, 16),\\
&\bar{a} - \bar{c} = (14, 35, 26), \qquad \bar{b} - \bar{d} = (13, 54, 26),\\
&\bar{a} - \bar{d} = (12, 43, 56), \qquad \bar{b} - \bar{e} = (12, 35, 46),\\
&\bar{a} - \bar{e} = (23, 45, 16), \qquad \bar{b} - \bar{f} = (14, 23, 56),\\
&\bar{a} - \bar{f} = (13, 52, 46), \qquad \bar{c} - \bar{d} = (15, 32, 46),
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\bar{c} - \bar{e} = (13, 24, 56),\\
&\bar{c} - \bar{f} = (12, 45, 36),\\
&\bar{d} - \bar{e} = (14, 52, 36),\\
&\bar{d} - \bar{f} = (24, 35, 16),\\
&\bar{e} - \bar{f} = (15, 34, 26),
\end{aligned}
$$

of which the first is assumed and the others deduced from it by applying the parallel generating permutations (28) can be shown to be consistent. For from the identities

$$
\begin{aligned}
&(153)(246) - (156)(234) + (136)(245) - (536)(241) = 0,\\
&(136)(425) - (134)(625) + (164)(325) - (364)(125) = 0,
\end{aligned}
$$

we get by addition the identity

$$(153)(246)-(156)(243)+(251)(346)-(256)(341)$$
$$+(142)(536)-(146)(532)=0,$$

whose terms pair off and give rise to

$$(33)\qquad \left\{\begin{matrix}123\\465\end{matrix}\right\}=(14,26,35)+(16,25,34)+(15,24,36)=0.$$

This identity written according to (32) has the form

$$(\bar a-\bar b)+(\bar b-\bar c)+(\bar c-\bar a)=0$$

and the consistency of (32) is verified throughout by the use of identities similar to (33). The six functions $\bar a,\cdots,\bar f$ can be obtained from the first column of (32) and the equation $\sum\bar a=0$.

If we subtract the two identities above we find that

$$\begin{aligned}2(136)(245)&=-(135)(264)+(134)(265)+(165)(234)\\&\quad-(164)(235)+(365)(124)-(364)(125)\\&=(13,26,45)+(16,23,54)+(36,12,54)\\&=(\bar b-\bar d)+(\bar e-\bar a)+(\bar f-\bar c)\\&=-2(\bar a+\bar c+\bar d)=2(\bar b+\bar e+\bar f).\end{aligned}$$

By permutation we obtain the ten relations

$$(34)\qquad\begin{aligned}&\bar d+\bar e+\bar f=-(123)(456), && \bar b+\bar d+\bar f=-(125)(436),\\&\bar c+\bar e+\bar f=-(146)(253), && \bar b+\bar d+\bar e=-(156)(234),\\&\bar c+\bar d+\bar f=-(134)(265), && \bar b+\bar c+\bar f=-(153)(246),\\&\bar c+\bar d+\bar e=-(126)(345), && \bar b+\bar c+\bar e=-(145)(326),\\&\bar b+\bar e+\bar f=-(136)(254), && \bar b+\bar c+\bar d=-(142)(356).\end{aligned}$$

Ten similar relations are obtained by using $\sum\bar a=0$.

Solving the equations (32) by using $\sum\bar a=0$ we find that

$$(35)\qquad\begin{aligned}6\bar a&=\quad(15,24,36)+(14,35,26)+(12,43,56)+(23,45,16)+(13,52,46),\\6\bar b&=-(15,24,36)\quad+(25,34,16)+(13,54,26)+(12,35,46)+(14,23,56),\\6\bar c&=-(14,35.26)-(25,34,16)\quad+(15,32,46)+(13,24,56)+(12,45,36),\\6\bar d&=-(12,43,56)-(13,54,26)-(15,32,46)\quad+(14,52,36)+(24,35,16),\\6\bar e&=-(23,45,16)-(12,35,46)-(13,24,56)-(14,52,36)\quad+(15,34,26),\\6\bar f&=-(13,52,46)-(14,23,56)-(12,45,36)-(24,35,16)-(15,34,26).\end{aligned}$$

It is to be noted that odd permutations of the points lead to odd permutations of $\bar{a}, \cdots, \bar{f}$ *without* change of sign and that $\bar{a}, \cdots, \bar{f}$ arise from $A, \cdots, F$ in (27) by replacing the product $(ij)(kl)(mn)$ in the order there written by (ij, kl, mn).

Consider the cubic $C = \bar{a}a + \cdots + \bar{f}f = 0$. It at most changes sign when the points are permuted; it is on the six points; and if it is on the meet of the lines $\overline{12}$ and $\overline{34}$ it must be on the 45 points of that kind, must contain the 15 lines (ikx), and therefore must vanish identically. From (31) and (29) we find that if $(12x) = (34x) = 0$, then

$$a + d = b + c = b + e = c + f = e + f = 0.$$

Then C takes the form

$$(\bar{a} - \bar{d})a + (\bar{b} + \bar{f} - \bar{c} - \bar{e})b = (\bar{a} - \bar{d})(a + b) + (\bar{b} + \bar{f} + \bar{d} - \bar{a} - \bar{c} - \bar{e})b.$$

If we substitute here from (34) and (32), replace x by $\overline{\overline{12}\overline{34}}$ by using (29) and (27), expand all terms like (ij, kl, mn) except $(12, 34, 56)$, and factor out $(123)(124)(134)(234)(125)(346)$, then C takes the form,

$$4(12, 34, 56) + 2[(56, 12, 34) - (126)(345) + (512)(634)] = 4(12, 43, 56) + 4(12, 34, 56) = 0.$$

Thus C vanishes identically and the required second identity connecting $a, \cdots, f$ is found.

(36) *If in the functions $A, \cdots, F$ as written in* (27) *each product*

$$(ij)(kl)(mn)$$

be replaced by the ternary determinant (ij, kl, mn), then the functions take values $6\bar{a}, \cdots, 6\bar{f}$ such that $\sum\bar{a} = 0$ and $\sum\bar{a}a = 0$.

The fact that cubic curves on the six points p of the plane π map the plane on a general cubic surface $C^{(3)}$ is well known but it seems that an explicit equation of $C^{(3)}$ in terms of the coördinates of the points p has not been given hitherto. Let us then derive the equations not only of $C^{(3)}$ but also of its 45 tritangent planes and therefore of its 27 lines. The six sets of directions about the points p map into skew lines $a_1, \cdots, a_6$ of $C^{(3)}$. The six conics on five of the six points map into six skew lines $b_1, \cdots, b_6$ of $C^{(3)}$. The lines a_i and b_i form a "double six," i. e., a_i meets b_j if $i \neq j$. The 15 lines (ijx) of π map into the remaining 15 lines c_{ij} of $C^{(3)}$. The 45 tritangent planes of $C^{(3)}$ separate into two sets with reference to the double six. Fifteen planes cut $C^{(3)}$ in the maps of cubic curves on π of the type $(ijx)(klx)(mnx)$; thirty in the maps of cubic curves made up of a line (ijx) and a conic on i, k, l, m, n.

As regards the equation of $C^{(3)}$ we have at once:

(37) *The equation of* $C^{(3)}$ *in Cremona's hexahedral form is* $\sum a^3 = 0$ *where the variables are subject to the relations,* $\sum a = 0$, *and* $\sum \bar{a}a = 0$.

From (31) there follows at once that 15 tritangent planes of $C^{(3)}$ are of the type $a + d = 0$, and that 15 lines of $C^{(3)}$ are of the type

$$a + d = b + e = c + f = 0.$$

The map of this particular line on π is $(12x) = 0$. On this line of $C^{(3)}$ there are five tritangent planes, three of which have just been given. The maps of the two others on π are the conics on 1, 3, 4, 5, 6 and on 2, 3, 4, 5, 6 each taken with the line $(12x) = 0$. One finds easily that the equations of the two maps are

$$\begin{aligned}(531)(461)(a+d) - (341)(561)(b+e) &= 0,\\ (532)(462)(a+d) - (342)(562)(b+e) &= 0.\end{aligned} \tag{38}$$

Since only products of the types $(ijk)(lmn)$ can be expressed in terms of $\bar{a}, \cdots, \bar{f}$ by means of (34) we must multiply the two cubics (38) together. By an easy manipulation of the term in $(a + d)(b + e)$ we get:

$$\begin{aligned}&(a+d)^2[(531)(462)\cdot(532)(461)]\\ &+(b+e)^2[(341)(562)\cdot(342)(561)]\\ &-(a+d)(b+e)[(531)(462)\cdot(341)(562)\\ &+(532)(461)\cdot(342)(561)+(563)(142)\cdot(512)(436)] = 0.\end{aligned} \tag{39}$$

Here the coefficient of $(a + d)(b + e)$ is worth noting. If we set

$$a_2 = \sum \bar{a}\bar{b}, \qquad a_3 = \sum \bar{a}\bar{b}\bar{c}, \qquad \cdots, \qquad a_6 = \bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}, \tag{40}$$

and then express this coefficient in terms of $\bar{a}, \cdots, \bar{f}$, it is

$$(\bar{b}+\bar{c}+\bar{f})(\bar{d}+\bar{c}+\bar{f}) - (\bar{c}+\bar{e}+\bar{f})(\bar{b}+\bar{d}+\bar{e}) - (\bar{b}+\bar{c}+\bar{d})(\bar{b}+\bar{d}+\bar{f}),$$

which takes the following form symmetric in $\bar{c}$ and $\bar{f}$:

$$(\bar{b}+\bar{c}+\bar{f})(\bar{d}+\bar{c}+\bar{f}) + (\bar{e}+\bar{c}+\bar{f})(\bar{a}+\bar{c}+\bar{f}) + \tfrac{1}{2}\{(\bar{b}+\bar{c}+\bar{d})(\bar{a}+\bar{c}+\bar{e}) + (\bar{b}+\bar{d}+\bar{f})(\bar{a}+\bar{e}+\bar{f})\}.$$

This by multiplication and use of $\sum \bar{a} = 0$ becomes

$$\overline{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \bar{c}\bar{f}). \tag{41}$$

Again by virtue of $\sum a = 0$ the quadratic (39) can be written in terms of $(a + d)^2$, $(b + e)^2$, $(c + f)^2$ and the coefficient of $(c + f)^2$ is half that of

$(a+d)(b+e)$ in (39). Hence (39) becomes

$$\overline{ad}(a+d)^2+\overline{be}(b+e)^2+\overline{cf}(c+f)^2=0. \tag{42}$$

The solution of (42) can be written in three ways,

$$\begin{aligned}(\overline{be}-d_2)(b+e)-(\overline{cf}+d_2)(c+f)&=0,\\ (\overline{cf}-d_2)(c+f)-(\overline{ad}+d_2)(a+d)&=0,\\ (\overline{ad}-d_2)(a+d)-(\overline{be}+d_2)(b+e)&=0,\end{aligned} \tag{43}$$

which are equivalent due to the fact that

$$-d_2^2=\overline{be}\,\overline{cf}+\overline{cf}\,\overline{ad}+\overline{ad}\,\overline{be}. \tag{44}$$

The two planes coincide if the irrationality d_2 vanishes. Then the conics on 1, 3, 4, 5, 6 and 2, 3, 4, 5, 6 must coincide and the six points must be on a conic. This condition, or rather its square, must be symmetric in the six points whence $d_2^2=\lambda a_2^2+\mu a_4$. If $\overline{be}\,\overline{cf}$ be multiplied out and symmetrized by using 45 terms we find that

$$d_2^2=a_2^2-4a_4. \tag{45}$$

In order to definitely fix the sign of d_2 and thereby isolate the members of the pair of planes in (42) let us set for brevity

$$\lambda=(531)(461),\qquad \mu=(532)(462),\qquad \nu=(341)(561),$$
$$\rho=(342)(562).$$

Then comparing (38) with (42) we find that

$$\tfrac{1}{2}(\overline{ad}+\overline{cf})=\lambda\mu,\qquad \tfrac{1}{2}(\overline{be}+\overline{cf})=\nu\rho,\qquad -\overline{cf}=\lambda\rho+\mu\nu. \tag{46}$$

Now $\mu\nu-\lambda\rho=0$ is clearly the condition that the six points be on a conic so that $\mu\nu-\lambda\rho=kd_2$, where d_2 is defined in (44). Squaring and using the values (46) we find that $k^2d_2^2=-\overline{be}\,\overline{cf}-\overline{cf}\,\overline{ad}-\overline{ad}\,\overline{be}$. Comparing this with (44) let us take $k=1$. Again $\overline{ad}=2\lambda\mu+\lambda\rho+\mu\nu$, $\overline{be}=2\nu\rho+\lambda\rho+\mu\nu$, $\overline{cf}=-\lambda\rho-\mu\nu$, $d_2=\mu\nu-\lambda\rho$, and (43) reduces to $\lambda(a+d)-\nu(b+e)$ which is the first of the planes in (38).

(47) *The* 45 *tritangent planes of* $C^{(3)}$ *comprise* 15 *of the type*

$$a+d=0$$

which maps into $(12x)(34x)(56x)$, *and* 30 *of the type given by* (43) *which maps into* $(12x)(13456x^2)$, *where the symbols* a_2, $\overline{cf}$, *and* d_2^2 *are defined by* (40), (41), (44), *and* (45), *while the sign of* d_2 *is defined by*

$$d_2=\begin{vmatrix}(341)(561) & (531)(461)\\ (342)(562) & (532)(462)\end{vmatrix}.$$

All the tritangent planes are deduced from these typical ones by the corresponding permutations (28) *of the letters and figures, an odd permutation being accompanied by a change of sign of* d_2.

5. The complete system of P_6^2

The P_6^2 is the simplest P_n^k such that neither P_n^k nor its associated Q_n^{n-k-2} can be given by a binary form. Thus there is a certain novelty in its complete system though a sketch of the results of these allied §§ 4, 5 has appeared in an earlier paper.*

By definition the general invariant I of P_6^2 is

$$I = \sum \pi (p_i p_k p_j)^{a_{ijk}}$$

where $\sum$ is symmetric in the six points and π contains each point to the degree ν.

The diophantine system contains six equations, $1°, \cdots, 6°$, the first of which reads as follows:

$1°$ $\quad a_{123} + a_{124} + a_{125} + a_{126} + a_{134} + a_{135} + a_{136} + a_{145} + a_{146} + a_{156} = \nu$.

When $\nu = 1$ we have ten solutions of the type

$$(a) \qquad (123)(456), \quad (124)(356), \quad \cdots.$$

When $\nu = 2$ we have thirty solutions of the type

$$(b) \qquad (123)(145)(256)(364).$$

Six of these contain (123), namely the six which occur in the terms of the determinant

$$(123)\begin{vmatrix} (156) & (164) & (145) \\ (256) & (264) & (245) \\ (356) & (364) & (345) \end{vmatrix}.$$

We shall now prove that every product π subject to the diophantine system can be expressed rationally and integrally in terms of the moduli (a) and (b). To prove this we have to show only that π contains a factor which can be so expressed. The remaining factor must then satisfy the system for a smaller value of ν and the proof is completed by exhaustion.

Let $a_{123} > 0$; if then $a_{456} > 0$, π contains a factor (a).

Let $a_{123} > 0$ and $a_{456} = 0$. By adding the first three and the last three of equations $1°, \cdots, 6°$ and by adding and subtracting the resulting equations we find that

$$\begin{aligned} a_{123} + \sum a_{rs,t} + \sum a_{mn,\sigma} &= 2\nu \\ 3a_{123} + \sum a_{rs,t} - \sum a_{mn,\sigma} &= 0 \end{aligned} \qquad \begin{pmatrix} rs = 23, 31, 12; & t = 4, 5, 6 \\ mn = 56, 64, 45; & \sigma = 1, 2, 3 \end{pmatrix}.$$

Hence $\sum a_{rs,t} = \nu - 2a_{123}$, $\sum a_{mn,\sigma} = \nu + a_{123}$.

*Johns Hopkins University Circular, no. 232 (1911), p. 59.

If none of the values $a_{mn,\sigma}$ which have the subscript $\sigma = 1$ are effective (i. e., greater than zero) then $\sum a_{1ik} < a_{123} + \sum a_{rs,t} = n - a_{123} < \nu$ which contradicts 1°. Hence of the effective values $a_{mn,\sigma}$ there is at least one for which $\sigma = 1$, and similarly at least one for which $\sigma = 2$, and at least one for which $\sigma = 3$.

If none of the values $a_{mn,\sigma}$ for which $mn = 56$ are effective then all the effective ones have $m = 4$ or $n = 4$. Hence $\sum a_{4ik} > \sum a_{mn,\sigma} > \nu$ which contradicts 4°. Thus of the effective values $a_{mn,\sigma}$ there is one at least for which $mn = 56$, one at least for which $mn = 64$, and one at least for which $mn = 45$.

If then there are only three effective values $a_{mn,\sigma}$ π must contain as a factor one term of the above determinant, i. e., a factor of type (*b*).

If there are four effective values $a_{mn,\sigma}$ and no term of the determinant is a factor then π must contain a factor of the type

$$(123)(156)(256)(364)(345).$$

Then unless $a_{642} = a_{542} = a_{415} = a_{416} = a_{423} = a_{413} = 0$, π contains a factor of type (*a*) or (*b*). Taking these to be zero, 4° becomes

$$\sum a_{4ik} = a_{412} + a_{435} + a_{436} = \nu.$$

If $a_{412} = 0$ then $a_{435} + a_{436} = \nu$ and $a_{123} + a_{435} + a_{436} > \nu$ which contradicts 3°. Hence $a_{412} > 0$ and π contains the factor

$$(123)(124)(345)(346)(561)(562)$$
$$= (124)(346)(561)(562)[(431)(253) + (234)(153)]$$

which is evidently reducible.

A fortiori if there are more than four effective values $a_{mn,\sigma}$, π is reducible. Hence every invariant I is a rational integral function of the moduli (*a*) and (*b*).

We see at once that the two moduli of type (*b*),

$$\alpha = (123)(145)(256)(364), \qquad \beta = (125)(143)(236)(564),$$

are closely related. In fact, $d_2 = \alpha - \beta = 0$ is the condition that the six points be on a conic. Thus the 30 moduli (*b*) divide into 15 pairs such that d_2 can be expressed as the difference of the members of a pair. Moreover

$$\begin{aligned}\alpha + \beta &= (256)(364)[(134)(215) + (142)(315)] \\ &\quad + (236)(564)[(154)(213) + (142)(513)] \\ &= (134)(256)\cdot(215)(364) + (236)(154)\cdot(564)(213) \\ &\quad + (142)(315)[(256)(364) - (236)(564)] \\ &= (134)(256)\cdot(215)(364) + (236)(154)\cdot(564)(213) \\ &\quad + (142)(365)\cdot(315)(246).\end{aligned}$$

Hence the thirty moduli (b) or the fifteen pairs α, β can be expressed rationally and integrally in terms of d_2 and the ten moduli (a). Moreover from (34) and (35) the ten moduli (a) and d_2^2 can be similarly expressed in terms of $\bar{a}, \cdots, \bar{f}$ whence

$$I = r(\bar{a}, \cdots, \bar{f}) + d_2 s(\bar{a}, \cdots, \bar{f})$$

where r and s are rational integral functions of their arguments. Applying the 360 even permutations of the points and adding the equations we find that

$$I = r_1(a_2, \cdots, a_6) + \sqrt{d}\, r_2(a_2, \cdots, a_6) + d_2 s_1(a_2, \cdots, a_6) + d_2\sqrt{d}\, s_2(a_2, \cdots, a_6),$$

where $a_2, \cdots, a_6$ are defined in (40) and

$$d = \pi(\bar{a} - \bar{b})^2. \tag{48}$$

Applying an even permutation we find that

$$I = r_1 - \sqrt{d}\, r_2 - d_2 s_1 + d_2\sqrt{d}\, s_2,$$

whence

$$2I = r_1(a_2, \cdots, a_6) + d_2\sqrt{d}\, s_2(a_2, \cdots, a_6).$$

(49) *The invariants* a_2, a_3, a_4, a_5, a_6, *and* $d_2\sqrt{d}$ *constitute a rational and integral complete system of invariants of* P_6^2. *There is but one syzygy connecting them—that which expresses the square of* $d_2\sqrt{d}$ *in terms of the others.*

Since $d_2^2 = a_2^2 - 4a_4$ the irrational invariants which are half symmetric can all be rationally expressed in terms of a_2, d_2, a_3, a_5, a_6, and $\sqrt{d}$.

The occurrence of the invariant $d_2\sqrt{d}$ in (49)—the vanishing of whose irrational factors d_2 and $\sqrt{d}$ have quite different projective interpretations—indicates that the $g_{6!}$ of permutations of the points is not the proper group to use in connection with P_6^2. One may use the $g_{\frac{1}{2}6!}$ whose invariants have just been given, or the $g_{2\cdot 6!}$ mentioned in § 10 whose invariants are $a_2, \cdots, a_6$.

6. Mapping of P_n^k's upon points of $\Sigma_{k(n-k-2)}$

In this paragraph various methods for mapping the class of projectively equivalent sets P_n^k upon a point P of a space $\Sigma_{k(n-k-2)}$ are discussed. One of the simplest type is selected and utilized later to study non-projective operations on the set P_n^k (such as a change in the order of its points) as operations on the point P.

A given point set P_n^k is projectively defined only when the points are ordered since it is in general self-projective only in the identical order. If there be given in S_k a basis, $(ub_1) = 0, \cdots, (ub_{k+2}) = 0$, consisting either of the reference points and the unit point (the "canonical basis") or of any other properly chosen $k + 2$ points, the set P_n^k can be linearly transformed so that

the first $k+2$ points, $(up_1), \cdots, (up_{k+2})$, fall in order upon $(ub_1), \cdots, (ub_{k+2})$. The remaining $n-k-2$ points then take positions, $(ux^{(1)}) = 0, \cdots, (ux^{(n-k-2)}) = 0$, which vary as P_n^k varies through projectively distinct classes.

In a space $\Sigma_{k(n-k-2)}$ choose a space $T^{(j)}_{k(n-k-3)-1}$ determined by the $k+1$ linear equations, $(\alpha_1^{(j)} y) = 0, \cdots, (\alpha_{k+1}^{(j)} y) = 0$. On $T^{(j)}$ there are ∞^k spaces $R^{(j)}_{k(n-k-3)}$ which can be put into one-to-one correspondence with the points of S_k. This is effected most simply by using the canonical basis in S_k and requiring that the particular $R^{(j)}$ defined by $(\alpha_i^{(j)} y) = 0$ $(i = 1, \cdots, l-1, l+1, \cdots, k+1)$ shall correspond to the reference point (ub_l) of S_k, and that the particular $R^{(j)}$ defined by $(\alpha_i^{(j)} y) = (\alpha_l^{(j)} y)$ $(i, l = 1, \cdots, k+1)$ shall correspond to the unit point of S_k. Then the equation of the $R^{(j)}_{k(n-k-3)}$ which corresponds to the point $(ux^{(j)}) = 0$ in S_k is

$$\begin{Vmatrix} (\alpha_1^{(j)} y) & (\alpha_2^{(j)} y) & \cdots & (\alpha_{k+1}^{(j)} y) \\ x_1^{(j)} & x_2^{(j)} & \cdots & x_{k+1}^{(j)} \end{Vmatrix} = 0 \tag{50}$$

$$(j = 1, \cdots, n-k-2).$$

Thus by choosing $n-k-2$ spaces $T^{(j)}$ in general position in $\Sigma_{k(n-k-2)}$ and by passing through each a space $R^{(j)}_{k(n-k-3)}$ determined by one of the last $n-k-2$ points of P_n^k we obtain a point P of Σ, the meet of the $n-k-2$ spaces $R^{(j)}$, which is the *map in* Σ *of the set* P_n^k *in* S_k. Conversely a general point P of Σ determines an $R^{(j)}$ on $T^{(j)}$ and therefore a point $(ux^{(j)}) = 0$ in S_k. These $n-k-2$ points together with the selected base is the P_n^k of S_k which corresponds to P in Σ. If the point P of Σ with coördinates y be given the coördinate $x_i^{(j)}$ of the point p_{k+2+j} of P^k is $(\alpha_i^{(j)} y)$ $(i = 1, \cdots, k+1; j = 1, \cdots, n-k-2)$. If the coördinates $x_i^{(j)}$ of P_n^k are given, the coördinates y of P are obtained by solving $k(n-k-2)$ independent linear equations obtained from the matrices (50).

We have supposed that $n \geq k+3$. If $n = k+3$ the above method of mapping fails. We have then merely to transform the first $k+2$ points of P_{k+3}^k to a selected basis and to take the last point as the map in $\Sigma_k = S_k$ of P_{k+3}^k. For the associated set Q_{k+3}^1 however the above construction is still valid unless $k = 1$.

The singular point sets P_n^k of this mapping are (1) those sets whose first $k+2$ points do not form a basis and (2) those sets for which the $n-k-2$ spaces $R^{(j)}$ meet in a line. These spaces meet in a line L if and only if the line L cuts each space $T^{(j)}$. That a line should meet a particular $T^{(j)}$ is k conditions whence there are $\infty^{k(n-k-2)-2}$ such lines which lie on a spread M in Σ. To determine the order of M let y be a point on such a line L and in Σ project from y upon a particular space say $T^{(1)}$ each of the other spaces $T^{(j)}$, i. e., pass an $R^{(j)}$ through y and $T^{(j)}$ cutting $T^{(1)}$ in an $S^{(1j)}_{k(n-k-3)-(k+1)}$. It is neces-

sary and sufficient that in $T^{(1)}$ the $n-k-3$ spaces $S^{(1j)}$ be on a point. This single condition is of order $n-k-3$ in y since in $T^{(1)}$ the coördinates of $S^{(1j)}$ are linear in y. For if $S^{(1j)}$ be required to meet a given S_{k-1} in $T^{(1)}$ in a point, then the S_{k-1} and $R^{(j)}$ meeting in a point determine in Σ an $S_{k(n-k-2)-1}$, the locus of possible positions of y. Hence the locus of lines L is a spread M^{n-k-3} of order $n-k-3$.

On the other hand the singular points P in Σ of the mapping are the points of the spaces $T^{(j)}$.

If we use a similar scheme of coördinate spaces $T'^{(j)}$ in a space Σ' (which may or may not be superposed upon Σ) in order to map the sets P_n^k, then the point y of Σ and the point y' of Σ' which map the same P_n^k are corresponding points in a Cremona transformation C. This transformation corresponds to a change of coördinates in the mapping. It is a particular case of the transformation obtained from bilinear forms. In fact its equations are

$$(51) \qquad \begin{Vmatrix} (\alpha_1^{(j)} y) & (\alpha_2^{(j)} y) & \cdots & (\alpha_{k+1}^{j} y) \\ (\alpha_1'^{(j)} y') & (\alpha_2'^{(j)} y') & \cdots & (\alpha_{k+1}'^{(j)} y') \end{Vmatrix} = 0$$

$$(j = 1, \cdots, n-k-2).$$

The order of such a transformation is in general $k(n-k-2)$ but due to the matrix form of the bilinear relations a reduction occurs here and the order of C is $n-k-2$. For as y runs over a line L in Σ the spaces $R^{(j)}$ are brought into projective relation. Hence in Σ' we get the locus of the intersections of corresponding members of $n-k-2$ projective pencils of $R'^{(j)}$'s. Each pencil lies in an $S'_{k(n-k+3)+1}$ and these have a common S'_{n-k-2}. In this common space we have $n-k-2$ projective pencils of S'_{n-k-3}'s which generate a rational norm-curve of order $n-k-2$, the correspondent of L, which meets each space $T'^{(j)}$ in $n-k-3$ points.

The singular points of C in Σ are (1) the points of the spaces $T^{(j)}$ and (2) the points of Σ for which the spaces $R^{(j)}$ are such that their corresponding spaces $R'^{(j)}$ meet in a line. For a particular point of $T^{(j)}$, $R'^{(j)}$ is indeterminate whence it corresponds to an S'_k. The locus of these S'_k's is a spread $N'^{(j)n-k-3}$ of dimension $k(n-k-2)-1$ and of order $n-k-3$ since a line L' meets it in the same number of points as the corresponding curve in Σ meets $T^{(j)}$. The points of the second kind lie on a singular manifold M of dimension $k(n-k-2)-2$ which correspond to the spread M'^{n-k-3} in Σ'. Hence

(52) *The order of the Cremona transformation C is $n-k-2$; its singular manifolds in Σ are M and the $n-k-2$ spaces $T^{(j)}$. The corresponding fundamental spreads in Σ' are all of order $n-k-3$.*

Thus an $S_{k(n-k-2)-1}$ in Σ is transformed by C into a spread of order $n-k-2$ in Σ' on the $n-k-1$ singular spreads in Σ'. This again is transformed by

C^{-1} into a spread of order $(n-k-2)^2$ in Σ from which there separates the $n-k-1$ fundamental spreads of order $n-k-3$ each, leaving the original linear spread.

For the general mapping described above the choice of $T^{(j)}$ implies

$$(k+1)k(n-k-3)$$

constants and the choice of the projectivity with S_k of spaces $R^{(j)}$ upon it implies $k(k+2)$ constants. Thus the mapping depends upon

$$k(n-k-2)(n-k-3)$$

absolute constants in Σ, a number which vanishes only when $n=k+3$. Now it is very desirable that the mapping apparatus should involve no absolute constants in Σ, both in the interest of simplicity and in order that the rôle played by the absolute constants of P_n^k itself be not obscured. For these reasons the following scheme, referred to hereafter as the "canonical mapping" seems most useful. In Σ choose an origin O and a space at ∞, $u=0$; and let the $k(n-k-2)$ rectangular coördinate spaces be divided into $n-k-2$ sets of k each say $y_i^{(j)}=0$ $(i=1,\cdots,k;\ j=1,\cdots,n-k-2)$. The space $T^{(j)}$ is then chosen to be $u=y_i^{(j)}=0$ $(i=1,\cdots,k)$. Thus all the spaces $T^{(j)}$ lie in the space at infinity. Each $T^{(j)}$ is a reference space in Σ whose skew reference space is an $S_k^{(j)}$. If the unit point in Σ be projected from $T^{(j)}$ upon the thus defined unit point of $S_k^{(j)}$ we obtain thereby a basis in $S_k^{(j)}$ which can be identified with the basis in S_k itself and thus the projectivity with S_k of spaces $R^{(j)}$ on $T^{(j)}$ is established. This mapping may be described analytically as follows:

(53) *If the set P_n^k in S_k be linearly transformed so that the first $k+2$ points fall upon the reference points and the unit point and if the coördinates of the remaining points be affected with such a factor of proportionality that the last coördinate of each is u, then the other $k(n-k-2)$ coördinates of these points together with u constitute the coördinates of the canonical map P in $\Sigma_{k(n-k-2)}$ of the set P_n^k in S_k. Two distinct canonical maps are equivalent under linear transformation.*

The coördinates of the point P in Σ are $y_i^{(j)}$, u where $y_i^{(j)}$ is the i-th coördinate of the $(k+2+j)$-th point of P_n^k after the set has been prepared as described in (53). Here $i=1,\cdots,k$ and $j=1,\cdots,n-k-2$, though it is sometimes convenient to let i take the value $k+1$ in which case $y_{k+1}^{(j)}=u$ $(j=1,\cdots,n-k-2)$.

If from the *first* $k+1$ points of P_n^k in order we drop the point p_r and adjoin from the remaining points of P_n^k the point p_s, the determinant of the resulting set of $k+1$ points will be denoted by $P(r;s)$; if from the *last* $n-k-1$ points of Q_n^{n-k-2} in order we drop the point q_r and prefix from the earlier points

of Q_n^{n-k-2} the point q_s, the determinant of the resulting set of $n-k-1$ points will be denoted by $Q(r;s)$. Then

(54) *The coördinates of the canonical map P of P_n^k in terms of the coördinates of the points of P_n^k are*

$$y_i^{(j)} = P(i;\, k+2+j)\cdot\delta_i\cdot\epsilon_j \qquad (i=1,\cdots,k+1;\ j=1,\cdots,n-k-2),$$

where

$$\delta_i = \prod_{l=1}^{k+1} P(l;\, k+2)/P(i;\, k+2),$$

$$\epsilon_j = \prod_{l=1}^{n-k-2} P(k+1;\, k+2+l)/P(k+1;\, k+2+j).$$

For the linear transformation which converts the first $k+2$ points of P_n^k into the canonical basis is

$$x_i' = P(i;\, x)/P(i;\, k+2) \quad \text{or} \quad x_i' = P(i;\, x)\cdot\delta_i \qquad (i=1,\cdots,k+1).$$

Then the last coördinate of the remaining points is

$$x_{k+1}'^{j} = P(k+1;\, k+2+j)\cdot\delta_{k+1} \quad (j=1,\cdots,n-k-2).$$

These are made equal by multiplication with ϵ_j respectively.

If the set P_n^k has been transformed into the canonical form and if all of the non-vanishing coördinates of the base points have been taken to be unity which we shall call the "prepared" form of P_n^k, then the determinants formed from them satisfy the following numerical relations:

$$P(k+1;\, k+2) = -P(k;\, k+2) = P(k-1;\, k+2) = \cdots = (-1)^k P(1;\, k+2) = 1,$$

(55) $$P(k+1;\, k+3) = P(k+1;\, k+4) = P(k+1;\, k+5) = \cdots = P(k+1;\, n) = u,$$

$$P(i;\, k+2+j) = (-1)^{k+i+1} y_i^{(j)} \qquad (i=1,\cdots,k \text{ or } k+1;\ j=1,\cdots,n-k-2).$$

The associated sets P_n^k and Q_n^{n-k-2} map into points P and P' in spaces $\Sigma_{k(n-k-2)}$ and $\Sigma'_{k(n-k-2)}$ respectively. Let us determine the Cremona transformation from P' to P. Let the sets be given in prepared form in which case the identity (2) is

$$\sum_{r=1}^{n} (up_r)\cdot\lambda_r(vq_r) = 0.$$

The determinants formed from the points p are proportional to those formed from the points q, the latter each being multiplied by the product of the λ's corresponding to the points of the determinant. Using a notation for such a

product similar to the notation for the determinant we have

$$P(k+1;\ k+2) = \epsilon \cdot \lambda(k+2;\ k+1) \cdot Q(k+2;\ k+1),$$
$$P(k;\ k+2) = -\epsilon \cdot \lambda(k+2;\ k) \cdot Q(k+2;\ k),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P(1;\ k+2) = (-1)^k \epsilon \cdot \lambda(k+2;\ 1) \cdot Q(k+2;\ 1).$$

Using the first of the relations (55) and the fact that

$$\lambda(k+2;\ k+1) : \lambda(k+2;\ k) : \cdots : \lambda(k+2;\ 1) = \lambda_{k+1} : \lambda_k : \cdots : \lambda_1$$

we find that

$$(a)\quad \lambda_1 : \lambda_2 : \cdots : \lambda_{k+1} = \frac{1}{Q(k+2;\ 1)} : \frac{1}{Q(k+2;\ 2)} : \cdots : \frac{1}{Q(k+2:\ k+1)}.$$

If we use similarly the second of relations (55) we get

$$(b)\quad \begin{aligned} u &= P(k+1;\ k+3) = -\epsilon \cdot \lambda(k+3;\ k+1) Q(k+3;\ k+1),\\ u &= P(k+1;\ k+4) = \epsilon \cdot \lambda(k+4;\ k+1) Q(k+4;\ k+1),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ u &= P(k+1;\ n) = (-1)^{n-k-2} \epsilon \cdot \lambda(n;\ k+1) Q(n;\ k+1), \end{aligned}$$

whence

$$(c)\quad \lambda_{k+3} : \lambda_{k+4} : \cdots : \lambda_n = Q(k+3;\ k+1) : (-1) Q(k+4;\ k+1) : \cdots : (-1)^{n-k-3} Q(n;\ k+1).$$

Finally from the last of relations (55) we have

$$(d)\quad (-1)^{k+i+1} y_i^{(j)} = P(i;\ k+2+j) = (-1)^{k+j+i+1} \epsilon \cdot \lambda(k+2+j;\ i) Q(k+2+j;\ i).$$

In (d) and (b) the products λ all contain λ_{k+2} to the first degree and this may be rejected as a factor of proportionality. They also each contain one factor from the set $\lambda_1, \cdots, \lambda_{k+1}$ and $n-k-3$ factors from the set $\lambda_{k+3}, \cdots, \lambda_n$ so that the factors of proportionality in (a) and (c) are not material. Dividing by the product of $\lambda_{k+3}, \cdots, \lambda_n$ we have finally that

(56) *The coördinates of the map P of the set P_n^k in terms of the coördinates of the map P' of the associated set Q_k^{n-k-2} are*

$$y_i^{(j)} = (-1)^{(j)} Q(k+2+j;\ i)/Q(k+2;\ i) Q(k+2+j;\ k+1) \qquad (i = 1, \cdots, k+1;\ j = 1, \cdots, n-k-2).$$

If this be cleared of fractions by multiplying by $\prod_i Q(k+2;\ i)$ and

$$\prod_j Q(k+2+j;\ k+1),$$

the right-hand members are of degree 1 in each of $q_1, \cdots, q_k$; of degree $n-k-2$ in each of q_{k+1}, q_{k+2}; and of degree $n-3$ in each of $q_{k+3}, \cdots, q_n$. The order of the transformation depends upon the norm $N_{n,k}$. If this be zero and $n = 2k+2$, the coördinates of $q_1, \cdots, q_{k+2}$ are numerical and the apparent order is $(2k-1)k$. But if a determinant Q contains neither q_{k+2} nor q_{k+1} it must contain the factor v. Thus $\Pi_i Q(k+2; i)$ contains the factor v^k while each denominator contains the factor v to at most the first power and the apparent order is reduced by $k-1$. Hence

(57) *The order in Σ_{k^2} of the involutory Cremona transformation between the maps of the associated sets P^k_{2k+2} and Q^k_{2k+2} is $k^2+(k-1)^2$.*

A simple form of two associated sets which shows the reciprocity between the two is

$$
\begin{array}{lllllllllll}
(1) & 1, & 0, & \cdots, & 0, & 0 & y_1^{(n-k-2)}, & y_1^{(n-k-3)}, & \cdots, & y_1^{(1)}, & u \\
(2) & 0, & 1, & \cdots, & 0, & 0 & y_2^{(n-k-2)}, & y_2^{(n-k-3)}, & \cdots, & y_2^{(1)}, & u \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(k) & 0, & 0, & \cdots, & 1, & 0 & y_k^{(n-k-2)}, & y_k^{(n-k-3)}, & \cdots, & y_k^{(1)}, & u \\
(k+1) & 0, & 0, & \cdots, & 0, & 1 & 1, & 1, & \cdots, & 1, & 1 \\
(k+2) & 1, & 1, & \cdots, & 1, & 1 & 0, & 0, & \cdots, & 0, & 1 \\
(k+3) & y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_k^{(1)}, & u & 0, & 0, & \cdots, & 1, & 0 \\
(k+4) & y_1^{(2)}, & y_2^{(2)}, & \cdots, & y_k^{(2)}, & u & 0, & 0, & \cdots, & 0, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(n-1) & y_1^{(n-k-3)}, & y_2^{(n-k-3)}, & \cdots, & y_k^{(n-k-3)}, & u & 0, & 1, & \cdots, & 0, & 0 \\
(n) & y_1^{(n-k-2)}, & y_2^{(n-k-2)}, & \cdots, & y_k^{(n-k-2)}, & u & 1, & 0, & \cdots, & 0, & 0
\end{array}
$$

One verifies at once that these sets satisfy an identity

$$\sum_{l=1}^{n} (up_l) \cdot \lambda_l (vq_l) \equiv 0$$

for $\lambda_1 = \cdots = \lambda_k = 1, \lambda_{k+1} = \lambda_{k+2} = -u, \lambda_{k+3} = \cdots = \lambda_n = -1$. They are not however simultaneously in the prepared form because of the inverted order of Q^{n-k-2}_n with reference to P^k_n.

7. The Cremona group of order $n!$ in $\Sigma_{k(n-k-2)}$ determined by P^k_n

In this paragraph we first consider the effect of changing the order of the points in P^k_n. If P is the map of P^k_n in $\Sigma_{k(n-k-2)}$ and if the permutation π is applied to the points of P^k_n, the map P' in Σ of P'^k_n, the permuted set, will be different from P unless P^k_n is self-projective according to the permutation π. But this is not in general the case when $n > k+2$. If P be given, the

prepared form of P_n^k is known (54). If π be applied to this prepared form and the resulting point set be prepared by proper linear transformation in S_k, then the point P' is determined. All of the operations involved in this process are rational whence P' and P correspond in a Cremona transformation of Σ. The $n!$ permutations π thus lead to $n!$ points P' including P itself. These $n!$ points have the property that their $n!$ sets $P_n'^k$ can all be superposed, by proper linear transformation in S_k, upon any definite one say P_n^k in some order. Thus P_n^k may be any one of the $n!$ sets and

(58) *The $n!$ points P' are a set of conjugate points under a Cremona group, $G_{n!}^k$, of order $n!$ in $\Sigma_{k(n-k-2)}$, which is isomorphic with the permutations π of the points in P_n^k.*

A set of generating permutations which leads to convenient generators of the Cremona group is made up of the transpositions $(l, l+1)$ $(l = 1, \cdots, n-1)$. These values of l divide into four sets, $1, \cdots, k$; $k+1$; $k+2$; $k+3, \cdots, n$, according to the form of the corresponding Cremona transformation. If the points p_l, p_{l+1} of the first set be interchanged, the linear transformation in S_k which restores the original basis is (x_l, x_{l+1}). If we note the effect of this upon the remaining $n-k-2$ points we find that the corresponding transformation in Σ is

$$(59)\qquad \begin{aligned} &(l, l+1): \qquad y_l'^{(j)} = y_{l+1}^{(j)}, \quad y_i'^{(j)} = y_i^{(j)} \\ &\qquad\qquad (i = 1, \cdots, k \text{ but } i \neq l, l+1), \\ &(l = 1, \cdots, k-1), \quad y_{l+1}'^{(j)} = y_l^{(j)} \qquad u' = u \\ &\qquad\qquad (j = 1, \cdots, n-k-2). \end{aligned}$$

The transposition $(k, k+1)$ interchanges the last two reference points. The linear transformation in S_k which restores them is (x_k, x_{k+1}). This destroys the equality of the last coördinates in the prepared form so that each of the new points $p_{k+3}, \cdots, p_n$ must be affected by a factor of proportionality which is the product of the last coördinates of the remaining ones. Hence the Cremona transformation in Σ is

$$(60)\qquad \begin{aligned} &(k, k+1): \quad y_m'^{(j)} = r_j\, y_m^{(j)}, \quad y_k'^{(j)} = r_j\, u, \quad u' = r, \\ &(m = 1, \cdots, k-1), \quad r = \prod_{m=1}^{n-k-2} y_k^{(m)}, \quad r_j = r/y_k^{(j)}. \end{aligned}$$

This transformation is of order $n-k-2$.

The transposition $(k+1, k+2)$ interchanges the last reference point and the unit point. The linear transformation in S_k which restores them is $x_l' = u - x_l$ $(l = 1, \cdots, k)$, $u' = u$. This gives rise to a linear transformation in Σ,

$$(61)\qquad (k+1, k+2): \quad y_i'^{(j)} = u - y_i^{(j)}, \quad u' = u \qquad (i = 1, \cdots, k;\ j = 1, \cdots, n-k-2).$$

The transposition $(k+2, k+3)$ interchanges the unit point and the point p_{k+3} whose coördinates are $x_l = y_l^{(1)}$ $(l = 1, \cdots, k)$ and $x_{k+1} = u$. The linear transformation which leaves the reference basis unaltered and transforms p_{k+3} into the unit point is $x_l' = x_l/y_l^{(1)}$, $x_{k+1}' = x_{k+1}/u$. It transforms the unit point into $1/y_1^{(1)}, \cdots, 1/y_k^{(1)}, 1/u$, and the point p_{k+2+j} into $y_1^{(j)}/y_1^{(1)}, \cdots, y_k^{(j)}/y_k^{(1)}$, 1 $(j = 2, \cdots, n-k-2)$. We must therefore multiply the coördinates of p_{k+3}' by u and then clear the coördinates of fractions. The resulting Cremona transformation is

$$(k+2, k+3): \quad y_l'^{(j)} = r_l y_l^{(j)}, \quad y_l'^{(1)} = u r_l, \quad u' = r$$

(62)
$$(j = 2, \cdots, n-k-2;\ l = 1, \cdots, k), \quad r = \prod_{l=1}^{k} y_l^{(1)},\ r_l = r/y_l^{(1)}.$$

This transformation is of order k.

For the last type of transposition the coördinates of the two points of P_n^k are merely interchanged so that the corresponding transformation in Σ is linear,

(63) $(l, l+1)$: $\quad y_i'^{(l)} = y_i^{(l+1)}, \quad y_i'^{(l+1)} = y_i^{(l)}, \quad y_i'^{(j)} = y_i^{(j)}, \quad u' = u$

$$(l = k+3, \cdots, n-1;\ \ i = 1, \cdots, k;\ \ j = 1, \cdots, n-k-2 \text{ but } j \neq l, l+1).$$

(64) *The Cremona $G_{n!}^k$ in $\Sigma_{k(n-k-2)}$ isomorphic with the $g_{n!}$ of permutations of P_n^k is generated by the elements* (59), $\cdots$, (63).

The $G_{n!}$ contains a collineation subgroup of order $2!\,k!\,(n-k-2)!$ generated by (59), (61), and (63). This is the direct product of the permutable groups of orders $k!$, $(n-k-2)!$, and $2!$ which correspond to the permutations of the columns, the permutations of the rows of the matrix $||y_i^{(j)}||$, and the involutory collineation which interchanges the origin and unit point of Σ leaving the other reference points unchanged. Thus the two points of the base in Σ form one conjugate set and the remaining base points form another conjugate set under collineations of $G_{n!}$. If $k = 1$ [$n-k-2 = 1$] the transformation (62) [(60)] is also a collineation so that all the transpositions except the first [last] lead to collineations which generate a collineation subgroup of order $(n-1)!$.

The remainder of this paragraph is devoted to the discussion of the invariants of $G_{n!}$ and their relation to the invariants of P_n^k as defined in § 3.

An invariant of a Cremona group is a form which is unaltered by each element of the group to within a factor composed of the fundamental spreads associated with the singular spreads of the particular element. Let

$$f_\rho(y_i^{(j)}, u) = \sum \Pi (y_i^{(j)})^{a_{ij}} u^\epsilon$$

be an invariant of $G_{n!}$ of order ρ. Let us consider the effect of the transformation $(k, k+1)$ upon f_ρ. This transformation has the $n-k-2$

singular manifolds $\Sigma^{(l)}$ $(l = 1, \cdots, n-k-2)$ defined by

$$y_i^{(j)} = u = 0 \quad (j = 1, \cdots, n-k-2;\ j \neq l;\ i = 1, \cdots, k),$$

whose corresponding fundamental spreads are $y_k^{(l)} = 0$; and the further singular manifold S defined by $y_k^{(j)} = 0$ $(j = 1, \cdots, n-k-2)$, whose fundamental spread is $u = 0$. If f_ρ contains a particular $\Sigma^{(l)}$ s times it must contain each $\Sigma^{(l)}$ s times since it is invariant under collineations which interchange the $\Sigma^{(l)}$. Suppose that it contains S also r times. Then it is transformed by $(k, k+1)$ into a spread of order $\rho(n-k-2)$ from which the fundamental spreads $y_k^{(j)}$ factor s times each and the fundamental spread u factors r times leaving the invariant factor f_ρ, i. e.,

$$s(n-k-2) + r = (n-k-3)\rho.$$

For a particular term $\pi (y_i^{(j)})^{\alpha_{i,j}} u^\epsilon$ let

$$\sum_{i,j} \alpha_{i,j} = \alpha = \rho - \epsilon$$

and let

$$\sum_{i=1}^{k} \alpha_{i,j} = \alpha_j.$$

Then since each term must contain $\Sigma^{(j)}$ at least s times we have that

$$\alpha + \epsilon - \alpha_1 = s + s_1, \qquad \alpha + \epsilon - \alpha_2 = s + s_2, \qquad \cdots,$$

$$\alpha + \epsilon - \alpha_{n-k-2} = s + s_{n-k-2},$$

where the s_j's are zero or positive. By addition,

$$(n-k-3)\alpha + (n-k-2)\epsilon = (n-k-2)s + \sum_j s_j \quad \text{or} \quad r + \epsilon = \sum_j s_j.$$

If then f_ρ be multiplied by u^r we get an invariant $f'_{\rho'}$ of order $\rho' = \rho + r$ which acquires under $(k, k+1)$ the factor

$$\prod_{j=1}^{n-k-2} (y_k^{(j)})^{s+r}.$$

In each term $\epsilon' = r + \epsilon = \Sigma s_j$ is such that $u^{\epsilon'}$ can be separated into the product $u^{s_1} u^{s_2} \cdots u^{s_{n-k-2}}$ where $\alpha_j + s_j = \alpha + \epsilon - s$. But $\alpha_j + s_j$ is the degree of f'_ρ in the coördinates of the $(k+2+j)$th point if u in u^{s_j} be regarded as the $(k+1)$th coördinate of this point. Hence

(65) *Every invariant of G_{n1} after multiplication by a proper power of u can be expressed as a form homogeneous and of the same degree ν in the coördinates of the $n-k-2$ points p_{k+2+j} $(j = 1, \cdots, n-k-2)$. The index of this power is in general the multiplicity of the singular manifold S on the invariant spread.*

The case of exception to the above value of the index is mentioned after (68).

We shall suppose hereafter that all invariants, i, of the Cremona group have been normalized as above by introducing a proper power of u. By using the formulæ of (54) the invariant i can be expressed in terms of the coördinates of the points of the general point set P_n^k. This expression must factor into an effective factor I depending upon the set and into an extraneous factor E depending upon the choice of the base. If for example the separation of $u^{\epsilon'}$ has been carried out as above the factors ϵ_j^{ν} from (54) will form a part of E and the factor I will then be of degree ν in the coördinates of each of the last $n-k-2$ points. Thus the vanishing of i implies that if all but one of the set P_n^k be fixed in S_k the remaining one must lie on a spread of order ν in S_k. Since as an invariant of $G_{n!}$ i has the same meaning irrespective of the permutations of the points, I must contain all the points to the same degree ν and contain them symmetrically. Moreover since the $y_i^{(j)}$ and u are invariants in the projective sense of P_n^k, I must be expressible in terms of determinants formed from the points of P_n^k. Hence I is an invariant of P_n^k as defined in § 3 and its degree ν must be such that $n\nu$ is divisible by $k+1$. All that has been said of spreads invariant under $G_{n!}$ applies equally well to invariant linear systems of spreads whence

(66) *Every normalized invariant spread or invariant linear system of spreads of $G_{n!}$ has in Σ the order $\nu(n-k-2)$ where $n\nu$ is divisible by $k+1$. When expressed in terms of the general set P_n^k it gives rise, to within an extraneous factor depending upon the base selected from P_n^k, to an invariant or linear system of irrational invariants of P_n^k of degree ν.*

That the degree ν of an invariant or irrational invariant is subject to no other condition than that given above is evident from the construction of the following simplest linear system of irrational invariants. Let D be the greatest common divisor of n and $k+1$, i. e., let $n = Dm$ and $k+1 = Dl$, where l and m are relatively prime. Arrange the n points of P_n^k in m sets of D each and from l of the m sets form a determinant of order $k+1$. By permuting the l sets cyclically through the m sets we get m determinants whose product is an irrational invariant of P_n^k of degree l, the lowest degree consistent with (66). The irrational invariants of this type give rise in Σ to an invariant linear system of spreads whose normalized order is $l(n-k-2)$. Hence

(67) *The linear system of spreads of lowest order in $\Sigma_{k(n-k-2)}$ invariant under $G_{n!}$ has the normalized order $l(n-k-2)$ where $k+1 = Dl$, D being the greatest common divisor of n and $k+1$.*

So far as the invariance of a spread in Σ under $G_{n!}$ is concerned the normalizing factor u^r is extraneous. Its value in the case above and indeed in general can be determined. If $N_{n,k} = n-2k-2 \leqq 0$, P_n^k can be arranged in the order $p_{k+1}, p_1, \cdots, p_k, p_{k+2}, \cdots, p_n$. Then the product of n deter-

minants, the first formed from the first $k+1$ points and the others formed from points succeeding these cyclically, is such that each determinant contains either p_{k+1} or p_{k+2} and therefore the corresponding spread in Σ does not contain the factor u. This product consists of D irrational invariants of the type (67) so that for this simplest linear system no normalizing factor is required. The simplest linear system for the associated Q_n^{n-k-2} is obtained by using the complementary determinants. But from the associated sets given on p. 182, § 6, it is clear that the values of the complementary determinants are the same to within factors u^s and the two linear systems coincide. But the normalized order of the second system as given by (67) is kl' where $n-k-1 = D'l'$, D' being the G.C.D. of n and $n-k-1$. Hence $D' = D$ and $l' = (n-k-1)/D$. Thus the difference between the actual order of the first system, $(n-k-2)(k+1)/D$ and the normalized order of the second, $k(n-k-1)/D$ is $r = (2k+2-n)/D = N_{n,\,n-k-2}/D$.

(68) *The normalized order of the invariant linear system* (67) *exceeds the actual order by* r *if* $N_{n,k} = Dr > 0$.

Since the order of any invariant is presumably a multiple of this lowest order the order of the normalizing factor is determined in general. However as the order of an invariant increases and a number of linearly independent invariants begin to appear, proper linear combinations can be formed so that u appears as a factor independently of the normalizing factor. This occurs when the invariant spread contains the singular manifold S of $(k, k+1)$ a larger number of times than is customary. Naturally in selecting the members of a complete system it would be convenient to choose those invariants which contain the highest possible power of u as a factor. This device is used in the discussion of P_6^2 in Part III of this paper.

The $G_{n!}$ associated with P_n^k appears as a natural generalization of Moore's "cross-ratio" group and we might expect to find in S_k some analogue of the cross-ratio of 4 points in S_1. If the points $p_3, \cdots, p_{k+2}$ lie in the S_{k-1}, $(\xi x) = (p_3, \cdots, p_{k+2}, x)$ and if the points $p_{k+3}, \cdots, p_{2k+2}$ lie in the S_{k-1}, $(\eta x) = (p_{k+3}, \cdots, p_{2k+2}, x)$, then

$$(69)\quad C = \frac{(\xi p_1)(\eta p_2)}{(\xi p_2)(\eta p_1)} = \frac{(3, 4, \cdots, k+2, 1)(k+3, \cdots, 2k+2, 2)}{(3, 4, \cdots, k+2, 2)(k+3, \cdots, 2k+2, 1)}.$$

is defined to be *a cross-ratio of the* $2k+2$ *points in* S_k; or more precisely *the cross-ratio of the pair* $1, 2$ *with respect to the sets* $3, \cdots, k+2$ *and* $k+3, \cdots, 2k+2$. It is in fact the ordinary cross-ratio on the line $\overline{12}$ of the points $1, 2$ with respect to the points where the line meets the two S_{k-1}'s. Its self-dual character is obvious. Moreover it is an absolute irrational invariant of the $2k+2$ points. Analogous irrational invariants of the same degree can be formed; for example when $k=3$ we have $\frac{(1234)(5678)}{(1256)(3478)}$. Since these irra-

tional invariants can be expressed by means of the above cross-ratios [thus $\frac{(1234)(5678)}{(1256)(3478)} = \frac{(1234)(5678)}{(1235)(4678)} : \frac{(1235)(4678)}{(1256)(3478)}$] and since they apparently are not binary cross-ratios determined by the $2k+2$ points it seems better to restrict the term as in (69).

The cross-ratios are formed from the ratios of members of the simplest invariant linear system determined by P^k_{2k+2}. The dimension of this linear system can be determined. For $k=1$ we have the three invariants $(12)(34)$, $(13)(42)$, $(14)(23)$, of which two are independent. For $k=2$ we have ten invariants like $(123)(456)$ of which according to (34) five are independent. For $k=3$, there are 35 invariants like $(1234)(5678)$ of which 14 are linearly independent. For each product in which 7 and 8 do not appear in the same factor is paired with another; e. g. $(ijk7)(lmn8)$ is paired with $(ijk8)(lmn7)$. The difference of the two can be expressed in terms of invariants like $(ijkl)(mn78)$. The sum of the two is the polar of the point pair 7, 8 as to the quadric $(ijkx)(lmnx)$. From the principle of translation just five of these quadrics are linearly independent. There are 15 invariants like $(ijkl)(mn78)$ but they are connected by 6 determinant identities. Moreover these 6 identities are linearly independent. For if the identities were linearly related then in this relation all terms containing the triad $v = \overline{ijk}$ within a determinant would cancel. But the identities contain $v = \overline{ijk}$ and $w = \overline{78}$ as follows:

$$(vl)(mnw) - (vm)(lnw) \cdots = 0, \qquad (vl)(mnw) - (vn)(lmw) \cdots = 0,$$
$$(vm)(lnw) - (vn)(lmw) \cdots = 0,$$

and these combinations are linearly independent. The identities then reduce the number 15 to 9 which with the 5 already obtained make up the number 14 of linearly independent invariants. The numbers 2, 5, 14 for $k = 1, 2, 3$ are given by $\binom{2k+2}{k+1}\frac{1}{k+2}$. Assuming this to be the true value up to k let us prove that it is true for k. Selecting as above two points we find (for the value $k-1$) $\binom{2k}{k}\frac{1}{k+1}$ independent polar forms leaving $\binom{2k}{k-1}$ products connected by $\binom{k+1}{k-2}$ identities. That the identities are independent is proven as above by taking k of the points to make up v and $k-3$ together with the two originally selected to make up w. Thus the number of linearly independent products is $\binom{2k}{k}\frac{1}{k+1} + \binom{2k}{k-1} - \binom{2k}{k-2} = \binom{2k+2}{k+1}\frac{1}{k+2}$. Hence

(70) *There are* $\binom{2k+2}{k+1}\frac{1}{k+2}$ *linearly independent determinant products of the type* $(1, 2, \cdots, k+1)(k+2, \cdots, 2k+2)$ *that can be formed from* $2k+2$ *points in* S_k. *The invariant linear system of lowest order,* (67), *under* $G_{(2k+2)!}$ *in* Σ_{k^2} *has the dimension* $\binom{2k+2}{k+1}\frac{1}{k+2} - 1$ *and the order* k.

Identities of higher degree connecting these products must exist when $k > 2$ since the dimension of the linear system is larger than the number of absolute constants in P^k_{2k+2}.

The cross-ratios of the set P^k_{2k+2} do not uniquely define the set when $k > 1$ since they take the same value for the associated set Q^k_{2k+2}. To within this ambiguity however they undoubtedly suffice to determine the set projectively.

If for P^k_n, $N_{n,k} > 0$ we can define a cross-ratio of the set P^k_n to be a cross-ratio of any $2k+2$ points of the set. Such a cross-ratio is an incomplete irrational absolute invariant of P^k_n. If then all the cross-ratios of P^k_n be given (subject of course to the algebraic relations connecting them) the set can be uniquely determined. For the ambiguity inherent in the determination of $2k+2$ points of P^k_n must be resolved when the conditions implied by the further cross-ratios are applied.

If for P^k_n, $N_{n,k} < 0$, then for the associated set Q^{n-k-2}_n, $N_{n,n-k-2} > 0$ and we can define the cross-ratios of P^k_n to be those of its associated set.

8. The form problem of P^1_5 and P^2_5

For $n = 5$ we have but two point sets, the associated P^1_5 and P^2_5. If we arrange their prepared forms in the following orders:

(1)	$1, 0$	y, x, u
(2)	$0, 1$	$1, 1, 1$
(3)	$1, 1$	$0, 0, 1$
(4)	x, u	$0, 1, 0$
(5)	y, u	$1, 0, 0$

the Cremona G_{51} in the space Σ_2 (variables x, y, u) has according to §7 the same form for both sets,

(71)

	(12)	(23)	(34)	(45)
x'	$= uy$	$= u - x$	$= u$	$= y$
y'	$= ux$	$= u - y$	$= y$	$= x$
u'	$= xy$	$= u$	$= x$	$= u$

The simplest invariant linear system is derived in the case of P^1_5 from products like $(12)(23)(34)(45)(51)$, and in the case of P^2_5 from products like $(345)(451)(512)(123)(234)$. In the first case according to (68) the products all contain the normalizing factor u. If this be removed the linear system in both cases is the system of ∞^5 cubic curves on the reference and unit points in Σ_2.

The invariants of G_{51} are merely the invariants of the binary quintic determined by P^1_5. They have been given from this point of view in $C1$ where also the form-problem of G_{51} has been set forth and solved by means of Klein's A-problem and ultimately in terms of the ikosahedral irrationality. The

solution there given is both direct and explicit. It may be worth while to point out here the rôle played in this solution by the above linear system of irrational invariants.

Instead of using the 12 cyclic irrational invariants let us replace them by the 10 irrational invariants like $(ij)^2(kl)(km)(lm)$ in terms of which the cyclic invariants can be expressed. They are connected by 5 linear relations of the type

$$(72)\quad (ij)^2(kl)(km)(lm) - (ik)^2(jl)(jm)(lm) + (il)^2(jk)(jm)(km) - (im)^2(jk)(jl)(kl) = 0.$$

The 5 relations are themselves linearly related so that there are 6 linearly independent invariants.

If we take a point set P_5^3 in S_3 which forms a base we can use a superfluous coördinate and write

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0.$$

A line π determined by points x and y has 10 coördinates connected by a set of 5 linear relations similar to (72),

$$(73)\quad \pi_{ij} + \pi_{ik} + \pi_{il} + \pi_{im} = 0 \qquad (i = 1, \cdots, 5;\ \pi_{ik} = -\pi_{ki}),$$

which also are linearly related so that 6 of the coördinates are linearly independent. A linear complex is given by an equation

$$(74)\quad \sum_{i,k} a_{ik}\pi_{ik} = 0 \qquad (i, k = 1, \cdots, 5;\ i \neq k;\ a_{ik} = -a_{ki}).$$

Since by virtue of (73) this complex can also be written as

$$\sum_{i,k} a'_{ik}\pi_{ik} = \sum_{i,k} (a_{ik} + \lambda_i)\pi_{ik} = 0,$$

the λ's can be so chosen that

$$(75)\quad a'_{ij} + a'_{ik} + a'_{il} + a'_{im} = 0 \qquad (i = 1, \cdots, 5).$$

Thus only 6 of the 10 complex coördinates are independent.

(76) *If we set* $a'_{ij} = (-1)^{\epsilon}(ij)^2(kl)(km)(kn)$ $(i < j)$, *where* ϵ *is odd or even according as the permutation ijklm is odd or even, then the point* x, y, u *of* Σ_2 *determines a linear complex in* S_3 *whose transformations by the collineation* $G_{5!}$ *of* P_5^3 *are isomorphic with the transformations of* x, y, u *in* Σ_2 *by the Cremona* $G_{5!}$.

This is the linear complex utilized in $C1$ to obtain the transition from the given form problem to Klein's A problem. That it is the linear complex determined by the twisted cubic on P_5^3 whose parameters are given by P_5^1 one can verify by a direct calculation. The analytic conditions which differentiate it from the general linear complex arise from the fact that the irrational

invariants employed satisfy five quadratic identities. The existence of these identities is evident from the fact that with each cyclic invariant,

$$\alpha = (12)(23)(34)(45)(51),$$

there is associated another,

$$\alpha' = (13)(35)(52)(24)(41),$$

such that $\alpha\alpha' = -\sqrt{D}$ where $\sqrt{D} = \prod(ik)$ $(i < k)$. Thus the 12 cyclic invariants are connected by 5 quadratic relations of the form

$$\alpha\alpha' = \beta\beta' = \cdots = \zeta\zeta'.$$

For the ten a'_{ij}'s there must exist five equivalent relations. The corresponding fact for the cubic curves in Σ_2 on a base is that they map the Σ_2 upon a quintic 2-way in S_5 which is the complete intersection of 5 quadrics.

In the solution of the quintic given in $C1$ the use of the Cremona group and its invariant system of cubic curves might have been avoided entirely by replacing the curves by the irrational invariants a'_{ij} of the quintic. Such a process however would be merely a circumlocution which would obscure the real nature of the case.

9. The parametric expression of a self-associated P_6^2 by means of theta-modular functions

Using the methods of finite geometry as set forth in $C3$ we identify the 15 rational period characteristics (Per. Char.) with denominator 2 of the theta function in two variables with the coördinates $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ of a point in the finite geometry modulo 2 in S_3. These points can be arranged in a basis configuration [$C3$, p. 271]

	B_1	B_2	B_3	B_4	B_5	B_6
B_1		$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
B_2	$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$		$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$
B_3	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$		$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$
B_4	$\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$		$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$
B_5	$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$		$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$
B_6	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$	

which is made up of 6 bases, each being the 5 points in a row or in a column. Since each point is in two bases and any two bases have a common point, a point can be denoted by P_{ik} $(i, k = 1, \cdots, 6; i \neq k)$ if it belongs to the bases B_i and B_k.

The 16 odd and even theta functions are identified with 16 quadrics in the finite geometry. The function $\vartheta(u) = \vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](u)$ is identified with the quadric $Q = x_1 x_3 + x_2 x_4 = 0$, and the function $\vartheta\left[\begin{smallmatrix}ij\\kl\end{smallmatrix}\right](u)$ with the quadric obtained by adding to Q the square of the polar plane of the point $\left(\begin{smallmatrix}ij\\kl\end{smallmatrix}\right)$ as to Q. These quadrics divide into 6 O quadrics Q_i $(i = 1, \cdots, 6)$ such that Q_i contains the five points P_{ik} of the base B_i; and into 10 E quadrics $Q_{ijk} = Q_{lmn}$ such that Q_{ijk} contains the 9 points P_{il}. A similar subscript notation is used for the corresponding theta functions. An even [odd] theta function does not [does] or does [does not] vanish for a given Per. Char. according as the corresponding quadric does or does not contain the corresponding point.

Any five theta squares are connected by a linear relation with constant coefficients. Let us find the relation connecting $\vartheta^2_{ijk}(u)$, $\vartheta^2_{ijl}(u)$, $\vartheta^2_{ijm}(u)$, $\vartheta^2_{ijn}(u)$, and say $\vartheta^2_k(u)$. If the Per. Char., P_{ij}, be substituted for u, the first four squares vanish while the last does not whence it cannot appear in the relation. Let us assume the relation to be

$$a_{ijk}\,\vartheta^2_{ijk}(u) + a_{ijl}\,\vartheta^2_{ijl}(u) + a_{ijm}\,\vartheta^2_{ijm}(u) + a_{ijn}\,\vartheta^2_{ijn}(u) = 0.$$

To determine the constants let u be P_{kl}, P_{km}, P_{kn} in turn. Then

$$a_{ijk}\,\vartheta^2_{ijk}(P_{kl}) + a_{ijl}\,\vartheta^2_{ijl}(P_{kl}) = 0,$$

$$a_{ijk}\,\vartheta^2_{ijk}(P_{km}) + a_{ijm}\,\vartheta^2_{ijm}(P_{km}) = 0,$$

$$a_{ijk}\,\vartheta^2_{ijk}(P_{kn}) + a_{ijn}\,\vartheta^2_{ijn}(P_{kn}) = 0.$$

If we use the formula*

$$\vartheta_{[\epsilon]_2}((u + \{\eta\}_2)) = e^{-\sum_{\mu=1}^{\mu=p}\sum_{\mu'=1}^{\mu'=p} a_{\mu\mu'}\frac{\eta_\mu \eta_{\mu'}}{4} - \sum_{\mu=1}^{\mu=p}\eta_\mu\left(u_\mu + \frac{\epsilon'_\mu + \eta'_\mu}{2}\pi i\right)}\vartheta_{[\epsilon+\eta]_2}((u)),$$

we find for $u = 0$ that

$$\vartheta^2_{[\epsilon]_2}((\{\eta\}_2)) = f_\eta \cdot e^{-\sum_{\mu=1}^{\mu=p}\eta_\mu \epsilon'_\mu}\vartheta^2_{[\epsilon+\eta]_2},$$

where $\vartheta_{ijk} = \vartheta_{ijk}(u)_{u=0}$ and f_η depends only on the Per. Char. $\{\eta\}_2$. Thus to within the factor f_η which disappears from the above equations we can set

$$\vartheta^2_{ijk}(P_{kl}) = \epsilon_{ijk,\,kl}\,\vartheta^2_{ijl},$$

where $\epsilon_{ijk,\,kl}$ is 1 or -1 according as $a\gamma + b\delta$ is congruent modulo 2 to 0 or 1

* Krazer, *Lehrbuch der Thetafunktionen*, p. 240, VII.

when $P_{kl} = \binom{ab}{cd}$ and $\vartheta_{ijk}(u) = \vartheta\left[\begin{smallmatrix}\alpha\beta\\ \gamma\delta\end{smallmatrix}\right](u)$. Hence to within sign $a_{ijk}, \cdots$, are proportional to $\vartheta^2_{ijk}, \cdots$. The signs of a_{ijk} and a_{ijl} are like or unlike according as $\epsilon_{ijk,\,kl}$ and $\epsilon_{ijl,\,kl}$ are like or unlike. But they are like or unlike according as their sum is even or odd. Since the sum of the characteristics ijk and ijl is the characteristic of P_{kl} we have only to test whether $ac + bd \equiv \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right.$ mod. 2, i. e., to test whether P_{kl} is or is not on the quadric $Q_{135} = Q_{246}$ associated with $\vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]$, i. e., whether $k + l$ is even or odd. Hence

(77) *The squares of the ten even thetas are connected by the* 15 *linear relations*

$$(-1)^k\,\vartheta^2_{ijk}\,\vartheta^2_{ijk}(u) + (-1)^l\,\vartheta^2_{ijl}\,\vartheta^2_{ijl}(u) + (-1)^m\,\vartheta^2_{ijm}\,\vartheta^2_{ijm}(u) + (-1)^n\,\vartheta^2_{ijn}\,\vartheta^2_{ijn}(u) = 0.$$

The ten determinant products formed from six points also are connected by (15) relations of the same kind. In any such identity two terms are obtained the one from the other by interchanging two letters and changing the sign; e. g., the pair of terms, $(-k-)(-l-) - (-l-)(-k-)$, occur in two identities. Denote by $\overline{(ijk)(lmn)}$ a determinant product such that in each determinant the natural order occurs. If this arrangement is made in the above schematic products where say $k < l$, and if r numbers lie between k and l, then in the first product they must actually lie between k and l so that in the second product r additional changes of sign are necessary to bring about the natural order. Taking account of the given change of sign and of the fact that $r + 1$ is even or odd according as $k + l$ is even or odd we see that

(78) *The ten determinant products formed from a* P^2_6 *are connected by* 15 *linear relations,*

$$(-1)^k\,\overline{(ijk)(lmn)} + (-1)^l\,\overline{(ijl)(kmn)} + (-1)^m\,\overline{(ijm)(kln)} + (-1)^n\,\overline{(ijn)(klm)} = 0,$$

where the dash indicates that the natural order is followed in each determinant.

Comparing the relations in (77) and (78) we see that it is permissible to set $\overline{(ijk)(lmn)} = \vartheta^2_{ijk}\,\vartheta^2_{ijk}(u)$; also for the value $u = 0$ to set $\overline{(ijk)(lmn)} = \vartheta^4_{ijk}$. In the latter case the P^2_6, being expressed in terms of the theta-modular functions of three moduli, must be subject to some condition. If in the theta relation (77) we set $u = P_{in}$ we get

$$(-1)^k\,\epsilon_{ijk,\,in}\,\vartheta^2_{ijk}\,\vartheta^2_{jkn} + (-1)^l\,\epsilon_{ijl,\,in}\,\vartheta^2_{ijl}\,\vartheta^2_{jln} + (-1)^m\,\epsilon_{ijm,\,in}\,\vartheta^2_{ijm}\,\vartheta^2_{jmn} = 0,$$

or

$$\sqrt{\vartheta^4_{ijk}\,\vartheta^4_{jkn}} + \sqrt{\vartheta^4_{ijl}\,\vartheta^4_{jln}} + \sqrt{\vartheta^4_{ijm}\,\vartheta^4_{jmn}} = 0.$$

This implies the following irrational condition on P^2_6:

$$\sqrt{\overline{(ijk)(lmn)}\cdot\overline{(jkn)(lmi)}} + \sqrt{\overline{(ijl)(kmn)}\cdot\overline{(jln)(kmi)}} + \sqrt{\overline{(ijm)(kln)}\cdot\overline{(jmn)(kli)}} = 0.$$

We have seen in § 3 that the condition that P_6^2 be on a conic can be given as

$$\sqrt{(lmn)(kji)\cdot(kjn)(lmi)} + \sqrt{(kmn)(lji)\cdot(ljn)(kmi)} + \sqrt{(kln)(mji)\cdot(mjn)(kli)} = 0.$$

To pass from the first to the second radicand k and l must be interchanged in two products whence from the above the numbers of changes of sign required in the radicands to restore the natural order within the determinants can differ only by even numbers and this condition reduces to the one given above. Hence the existing condition is that P_6^2 be on a conic. To identify the six points on the conic we note that if the conic be taken parametrically the ternary determinant product $(ijk)(lmn)$ becomes the binary determinant product $(ij)(ik)(jk)\cdot(lm)(ln)(mn)$. According to Bolza* this product is $\rho^2\,\vartheta_{ijk}^4$ whence

(79) *If the fourth powers of the ten even thetas for the zero values of the arguments be equated to the ten determinant products formed from P_6^2, i. e., if $\vartheta_{ijk}^4 = (ijk)(lmn)$, they define a P_6^2 on a conic whose parameters determine the fundamental binary sextic (P_6^1) which is associated with the normal hyperelliptic integrals of the first kind whose canonical periods are the moduli τ_{11}, τ_{12}, τ_{22} of the theta functions.*

If the ten products be expressed in terms of $\bar{a}, \cdots, \bar{f}$ as in § 4 we have found that the condition that P_6^2 be on a conic is $a_2^2 - 4a_4 = 0$. This is, in S_4 (variables $\bar{a}, \cdots, \bar{f}$), the quartic spread of Maschke,† the reciprocal of the cubic spread of Richmond,‡ whence we find anew an implicit result due to Maschke:

(80) *The quartic spread, $a_2^2 - 4a_4 = 0$, in S_4 whose reciprocal is $\sum_{i=1}^{i=6} u_i^3 = 0$ where $\sum_{i=1}^{i=6} u_i = 0$, can be expressed parametrically and uniformly in terms of the theta-modular functions of genus two.*

Explicit expressions for the coördinates of the P_6^2 in terms of the determinant products and therefore of the ϑ_{ijk}^4's are given below in (89) and (90).

10. The form problems of P_6^1, P_6^2, and P_6^3

For $n = 6$ we have three point sets of which that in S_1 and that in S_3 are associated. If we take the prepared forms of these in the order

* Mathematische Annalen, vol. 30 (1887), p. 485; see also the references given there. The relations implied by (79) also are found on pp. 493–4. Indeed the relations (77) are well-known in the theory. They are derived afresh here in order to reconcile the notation for the thetas with that for the points and to identify the signs without comparing lists of particular formulæ.

† Mathematische Annalen, vol. 30 (1887), p. 496; in particular p. 505.

‡ For references see *C*2:

(81)

(1)	1, 0	t, s, r, u
(2)	0, 1	1, 1, 1, 1
(3)	1, 1	0, 0, 0, 1
(4)	r, u	0, 0, 1, 0
(5)	s, u	0, 1, 0, 0
(6)	t, u	1, 0, 0, 0

the Cremona $G_{6!}$ in the space Σ_3 (variables r, s, t, u) for both sets is generated by

(82)

	(12)	(23)	(34)	(45)	(56)
r'	$= ust$	$= u - r$	$= u$	$= s$	$= r$
s'	$= utr$	$= u - s$	$= s$	$= r$	$= t$
t'	$= urs$	$= u - t$	$= t$	$= t$	$= s$
u'	$= rst$	$= u$	$= r$	$= u$	$= u$

The simplest invariant linear system for P_6^1 is obtained from the irrational invariants like (12)(34)(56) all of which contain the normalizing factor u; that for P_6^3 is obtained from irrational invariants like (3456)(5612)(1234). In both cases the linear system is the system of quadrics on the canonical basis in Σ_3. The invariants of P_6^1 have been obtained in § 3; those of P_6^3 have been inferred from the invariants of P_6^1 in *C*2 § 1 where the form problem of $G_{6!}$ in Σ_3 is set forth. This form problem is solved by using the resolvent sextic of Maschke (loc. cit.). If in connection with P_6^1 and P_6^3 we introduce the self-associated P_6^2 Maschke's sextic and its significance appear at once.

Taking P_6^2 in the prepared form

(83)

(1)	1, 0, 0	(4)	1, 1, 1
(2)	0, 1, 0	(5)	x, y, u
(3)	0, 0, 1	(6)	z, t, u

we have the following equations for the generators of $G_{6!}^2$ in Σ_4 (variables x, y, z, t, u)

(84)

	(12)	(23)	(34)	(45)	(56)
x'	$= y$	$= xt$	$= u - x$	$= uy$	$= z$
y'	$= x$	$= ut$	$= u - y$	$= ux$	$= t$
z'	$= t$	$= zy$	$= u - z$	$= zy$	$= x$
t'	$= z$	$= uy$	$= u - t$	$= tx$	$= y$
u'	$= u$	$= yt$	$= u$	$= xy$	$= u$

The simplest invariant linear system is obtained from the irrational invariants like (123)(456). If we express these in terms of $\bar{a}, \cdots, \bar{f}$ we find that

$$
\begin{aligned}
3\bar{a} &= \rho - 3(ux + ut), \qquad 3\bar{d} = \rho - 3(uy + uz),\\
3\bar{b} &= \rho - 3(ux + yz), \qquad 3\bar{e} = \rho - 3(uy + xt),\\
3\bar{c} &= \rho - 3(ut + yz), \qquad 3\bar{f} = \rho - 3(uz + xt),\\
&\rho = u(x + y + z + t) + xt + yz.
\end{aligned} \tag{85}
$$

Thus the invariant linear system can be expressed in terms of 5 quadrics on the canonical basis in Σ_4, which contain the four lines in the Σ_3, $u = 0$, whose equations are $x = y = 0$, $x = z = 0$, $t = y = 0$, $t = z = 0$. The simplest members of the system are the differences of the six terms in ρ. Four general quadrics of the system having a skew quadrilateral and two further points in common will meet in two additional points $x, \cdots, u$ and $x', \cdots, u'$. Either one of these two points determines the other and the quadrics take proportional values for the two points. This is true of the maps P and P' of two associated P_6^2's whence the two points are corresponding points in the Cremona transformation B of order 5 whose equations are given in (56). The transformation B together with $G_{6!}^2$ determine in Σ_4 a $G_{2\cdot 6!}$ which corresponds in S_2 to the permutations of the points of the associated P_6^2 and Q_6^2 and to the interchange of the sets. The element B is permutable with each element of $G_{6!}^2$ and $G_{2\cdot 6!}$ is the product of the groups $G_{6!}$ and the G_2 with elements B and $B^2 = 1$. The invariants of $G_{6!}$ and $G_{2\cdot 6!}$ have been obtained in § 5.

The form problem of $G_{6!}^2$ is as follows:

(86) *Given the values of* a_2, a_3, a_4, a_5, a_6, *and* $d_2\sqrt{d}$, *subject to the syzygy which connects* $(d_2\sqrt{d})^2$ *with* $a_2, \cdots, a_6$ *to find the* 720 *points in* Σ_4 *for which these invariants take the assigned values.*

Here

$$
d_2 = \sqrt{a_2^2 - 4a_4} = \begin{vmatrix} (341)(561) & (531)(461) \\ (342)(562) & (532)(462) \end{vmatrix}
$$

$$
= u[yz(x + t - u) - xt(y + z - u)];
$$

while $d = \prod(\bar{d} - \bar{a})^2 = \prod(12, 34, 56)^2 = \prod(u[x + t - z - y])^2$. If $d_2 = 0$, P_6^2 is on a conic; if $d = 0$, P_6^2 is on three lines which meet in a point. Let us first verify that the given values determine 720 points. From $a_2, \cdots, a_6$ there is derived, by using the permutations of the roots of the sextic with these coefficients, 720 value systems $\bar{a}, \cdots, \bar{f}$. To each of these systems there correspond two points. Under the even subgroup $G_{\frac{1}{2}6!}$ of $G_{6!}$, which is invariant under $G_{2\cdot 6!}$ these 1,440 points divide into four conjugate sets of 360 each, say S_0, S_1, S_0', S_1'. If π is an odd permutation of $G_{6!}$ the elements of $G_{2\cdot 6!}$ divide with regard to $G_{\frac{1}{2}6!}$ into four partitions, namely, $G_{\frac{1}{2}6!}$, $G_{\frac{1}{2}6!}\pi$, $G_{\frac{1}{2}6!}B$, $G_{\frac{1}{2}6!}\pi B$. These partitions permute the four sets above as follows: $(S_0)(S_1)(S_0')(S_1')$, $(S_0S_1)(S_0'S_1')$, $(S_0S_0')(S_1S_1')$, $(S_0S_1')(S_1S_0')$. The

element π effects not only a change of sign in $\sqrt{d}$ but also a change of sign in d_2. The element B effects a change of sign in d_2 only. Thus only the two sets S_0, S_1 arise from the given values while for the sets S_0', S_1' the invariant $d_2\sqrt{d}$ takes the opposite sign. We have already remarked (§ 1) that the sets P_6^2 and Q_6^2 can coincide in two ways if only one condition be imposed. They can coincide in the identical order if $d_2 = 0$, and in an odd order like $(12)(34)(56)$ if $\bar{a} = \bar{d}$. Thus as we should expect S_0', S_1' coincide with S_0, S_1 if either $d_2 = 0$ or $d = 0$.

To solve the given form problem we adjoin $\sqrt{d}$ for the known sextic. Then both d_2 and $\sqrt{d}$ are rationally known, and the group of the problem is reduced to G_{1461}. From one of the resulting 360 value systems of $\bar{a}, \cdots, \bar{f}$ we get from (85)

$$
\begin{aligned}
6yz &= \rho + 3(\bar{a} - \bar{b} - \bar{c}), & 6xt &= \rho + 3(\bar{d} - \bar{e} - \bar{f}),\\
6ut &= \rho + 3(-\bar{a} + \bar{b} - \bar{c}), & 6uz &= \rho + 3(-\bar{d} + \bar{e} - \bar{f}),\\
6ux &= \rho + 3(-\bar{a} - \bar{b} + \bar{c}), & 6uy &= \rho + 3(-\bar{d} - \bar{e} + \bar{f}).
\end{aligned} \tag{87}
$$

The product of the left-hand members in each of the two columns are equal. This fact applied to the right-hand members yields a quadratic equation in ρ. If we set

$$
\begin{aligned}
\sigma_1 &= \bar{a} + \bar{b} + \bar{c}, & \tau_1 &= \bar{d} + \bar{e} + \bar{f},\\
\sigma_2 &= \bar{b}\bar{c} + \bar{c}\bar{a} + \bar{a}\bar{b}, & \tau_2 &= \bar{e}\bar{f} + \bar{f}\bar{d} + \bar{d}\bar{e},\\
\sigma_3 &= \bar{a}\bar{b}\bar{c}, & \tau_3 &= \bar{d}\bar{e}\bar{f},
\end{aligned}
$$

whence

$$
a_2 = \sigma_2 + \tau_2 + \sigma_1\tau_1, \qquad a_4 = \sigma_3\tau_1 + \sigma_1\tau_3 + \sigma_2\tau_2,
$$

this quadratic equation is

$$
\begin{aligned}
\rho^2(\sigma_1 - \tau_1) - 12\rho(\sigma_2 - \tau_2) + 9[(\sigma_1 - \tau_1)\sigma_1\tau_1 \\
+ 4(\sigma_1\sigma_2 - \tau_1\tau_2) - 8(\sigma_3 - \tau_3)] = 0.
\end{aligned}
$$

Its discriminant easily reduces on using $\sigma_1 + \tau_1 = 0$ to $36(a_2^2 - 4a_4) = 36d_2^2$. Hence

$$
\rho = 6\frac{(\sigma_2 - \tau_2) + d_2}{(\sigma_1 - \tau_1)}. \tag{88}
$$

Since d_2 and therefore ρ is now rationally known we have

$$
\begin{aligned}
6u \cdot x &= \rho + 3(-\bar{a} - \bar{b} + \bar{c}),\\
6u \cdot y &= \rho + 3(-\bar{d} - \bar{e} + \bar{f}),\\
6u \cdot z &= \rho + 3(-\bar{d} + \bar{e} - \bar{f}),\\
6u \cdot t &= \rho + 3(-\bar{a} + \bar{b} - \bar{c}),\\
6u \cdot u &= \frac{6uz \cdot 6uy}{6yz} = \frac{[\rho + 3(-\bar{d} + \bar{e} - \bar{f})][\rho + 3(-\bar{d} - \bar{e} + \bar{f})]}{[\rho + 3(\bar{a} - \bar{b} - \bar{c})]}.
\end{aligned} \tag{89}
$$

This determines the point x, y, z, t, u in Σ_4 which is a solution of (86). The

720 points can be obtained from the one either by effecting the operations of G_{61} or by permuting $\bar{a}, \cdots, \bar{f}$ in (89) provided the sign of d_2 be changed for an odd permutation. Another expression for the coördinates can be obtained from (89) which introduces the coefficients of the tritangent planes of C^3 as given in (47). This is

$$
\begin{aligned}
&\lambda x = d_2 - \overline{ab}, \qquad \lambda y = d_2 + \overline{de}, \\
&\lambda z = d_2 + \overline{df}, \qquad \lambda t = d_2 - \overline{ac}, \\
&\lambda u = \frac{(d_2 + \overline{de})(d_2 + \overline{df})}{(d_2 - \overline{bc})}, \qquad \lambda = (\sigma_1 - \tau_1)u.
\end{aligned} \tag{90}
$$

The Cremona G_{61} furnishes an expression of the algebraic relations which must exist among these coefficients.

The above solution of the form problem of the general P_6^2 is based on the solution of the sextic. On the other hand the form problem of the particular self-associated P_6^2 is of vital importance in the solution of the sextic itself. The sextic S defines a P_6^1. After the adjunction of $\sqrt{D}$, S has a rational sextic resolvent Σ, the sextic of Joubert, which is suggested at once by P_6^3. These two point sets—the one on a line and the other on a twisted cubic—are evidently related to the P_6^2 on a conic. And indeed this P_6^2 determines the sextic Φ of Maschke whose roots can be identified with $\bar{a}, \cdots, \bar{f}$. According to $C2$, p. 323 (32), the roots of Σ can be rationally expressed in terms of the roots of Φ and in turn the roots of S in terms of the roots of Σ. Moreover the solution of Φ in terms of hyperelliptic modular functions is apparent from the last paragraph. Thus the solution of the general sextic S breaks into two parts.

The first or transcendental part of the solution consists of the following operations: From the periods of a pair of normal hyperelliptic integrals of the first kind attached to the Riemann surface $y^2 = S$, the moduli $\tau_{11}, \tau_{12}, \tau_{22}$ of the theta functions are obtained. Then from the well-known series which define them the values of the ten ϑ_{ijk}^4's (or of any five linearly independent ones) are calculated.

The second or algebraic part is as follows: From the ten ϑ_{ijk}^4's the determinant products $(ijk)(lmn)$ of the P_6^2 are obtained in (79) according to which this P_6^2 is parametrically projective to S. After finding $\bar{a}, \cdots, \bar{f}$ from the $(ijk)(lmn)$, either we can use (89) or (90) to determine the P_6^2 and can find the parameters of its six points on the conic, or having the roots of Σ we can get from $C2$, p. 319 (19), a solution of the form problem of P_6^3. In either case we have the roots of a sextic projective to S and can obtain therefrom, by using the transformation, $C2$, p. 318 (17), the roots of S itself. It is worth noting that in this algebraic part no accessory irrationalities are required.

BALTIMORE,
December 1, 1914.

SCROLL DIRECTRIX CURVES*

BY

CHAS. T. SULLIVAN

In his theory of surfaces based on the discussion of a non-involutory completely integrable system of two partial differential equations of the second order, Professor Wilczynski has shown that the organic curves which he calls the *directrix curves* play an important rôle.† They are defined in terms of the osculating linear complexes of the asymptotic curves. When the surface is a *scroll*, the straight line generators constitute one of the families of its asymptotic lines and any one of these asymptotic lines is contained in a quadruply infinite family of linear complexes. Thus the osculating linear complex becomes indeterminate and, therefore, the *directrix curves* cease to be defined. The purpose of the present paper is to define and discuss a set of curves on a ruled surface which shall correspond as closely as possible to the *directrix curves* of a non-ruled surface. We shall call them *scroll directrix curves*, and we shall study certain configurations involving these curves and their associated *line congruences*. The results obtained in this paper will constitute the counterpart for *scrolls* of certain theorems previously established for non-ruled surfaces.

Definition of the linear complex $C''(u)$

Since the integrating surface S is to be a *scroll*, we may assume that the system of partial differential equations has been reduced to the following canonical form:‡

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + gy = 0, \tag{1}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.}$$

The coefficients of these equations are the fundamental seminvariants of the system. The curves u = const. will be the straight line generators, and the

* Presented to the Society, Chicago, December, 1913.

† Wilczynski, *Differential Geometry of Curved Surfaces*, these Transactions, vols. 8, 9, and 10. These memoirs will be cited as M_1, M_2, M_3.

‡ M_3, pp. 293, 294.

curves $v = \text{const.}$ will be the twisted asymptotic curves on the integrating surface S.*

The integrability conditions of (1) are

$$(2) \qquad g_u = 0, \qquad b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0.$$

The fundamental semi-covariants

$$y, \qquad z = y_u, \qquad \rho = y_v, \qquad \sigma = y_{uv}$$

determine the tetrahedron $P_y P_z P_\rho P_\sigma$ and a corresponding system of coördinates, in which the coördinates of the point $\alpha y + \beta z + \gamma\rho + \delta\sigma$ are defined to be $(\alpha, \beta, \gamma, \delta)$.

The linear complex $C'(v)$ osculating the asymptotic curve Γ' at the point P_y, when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, has the equation†

$$(3) \qquad -b_v \omega_{34} - b\omega_{14} + b\omega_{23} = 0.$$

We shall now define a linear complex $C''(u)$ to take the place of the osculating linear complex of the asymptotic curve Γ'', which is indeterminate since Γ'' is a straight line. The linear complex C'' shall contain the asymptotic line Γ''. It shall be in involution with the complex C' which osculates the asymptotic curve Γ' at the point P_y. The null-plane of the point P_y in the complex C'' shall be the tangent plane of the surface S at this point. The *unode* and *singular tangent plane* of the *Cayley cubic scroll* osculating the integrating surface S at the point P_y shall be incident elements in the null-system of C''.

Let $\sum a_{ij}\omega_{ij} = 0$ be the equation of a linear complex referred to $P_y P_z P_\rho P_\sigma$. The first two of the above conditions will be satisfied at the point P_y if

$$a_{12} = a_{13} = 0 \quad \text{and} \quad a_{14} = a_{23}.$$

Hence the complex C'' is contained in the family of complexes

$$\omega_{14} + \omega_{23} + \alpha_{34}\omega_{34} + \alpha_{42}\omega_{42} = 0.$$

Now consider the point P_y displaced to the point $P_{y+y_v\delta v}$ along Γ''. The semi-covariants y, z, ρ, σ will become $\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma}$, respectively, where

$$\bar{y} = y + \rho dv + \cdots$$

$$\bar{z} = z + \sigma dv + \cdots$$

$$\bar{\rho} = \rho - gydv + \cdots$$

$$\bar{\sigma} = \sigma - gzdv + \cdots$$

* We shall hereafter refer to the curves $v = \text{const.}$ as Γ', and to the curves $u = \text{const.}$ as Γ''.

† M_2, p. 92.

Let the coördinates of a point Q be $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ when referred to $P_{\bar{y}}\, P_{\bar{z}}\, P_{\bar{\rho}}\, P_{\bar{\sigma}}$ and (x_1, x_2, x_3, x_4) when referred to $P_y\, P_z\, P_\rho\, P_\sigma$. Then

$$(\bar{x}_1 \bar{y} + \bar{x}_2 \bar{z} + \bar{x}_3 \bar{\rho} + \bar{x}_4 \bar{\sigma})$$

must be identically equal to $(x_1 y + x_2 z + x_3 \rho + x_4 \sigma)$, apart from a factor of proportionality. Hence, if we neglect infinitesimals of the second and higher orders, we find the following equations of transformation between the coördinate systems $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ and (x_1, x_2, x_3, x_4):

$$\omega x_1 = \bar{x}_1 - g\bar{x}_3\, dv, \qquad \omega x_2 = \bar{x}_2 - g\bar{x}_4\, dv,$$
$$\omega x_3 = \bar{x}_3 + \bar{x}_1\, dv, \qquad \omega x_4 = \bar{x}_4 + \bar{x}_2\, dv,$$

and

$$\omega' \bar{x}_1 = x_1 + g x_3\, dv, \qquad \omega' \bar{x}_2 = x_2 + g x_4\, dv,$$
$$\omega' \bar{x}_3 = x_3 - x_1\, dv, \qquad \omega' \bar{x}_4 = x_4 - x_2\, dv,$$

where ω and ω' are factors of proportionality. From these we find

$$\begin{aligned} \bar{\omega}_{14} &= \omega_{14} - (\omega_{12} - g\omega_{34})\, dv, \\ \bar{\omega}_{23} &= \omega_{23} + (\omega_{12} - g\omega_{34})\, dv, \\ \bar{\omega}_{34} &= \omega_{34} + (\omega_{23} - \omega_{14})\, dv, \\ \bar{\omega}_{42} &= \omega_{42}. \end{aligned} \tag{4}$$

The canonical coördinates of the *unode* of the Cayley cubic scroll osculating the integrating surface S at the point P_y are $(0, 0, 1, 0)$.* The canonical coördinates of a point are connected with (x_1, x_2, x_3, x_4) by the equations

$$X = \lambda(x_2 + \beta x_4), \qquad Y = \mu(x_3 + \alpha x_4),$$
$$Z = \lambda\mu x_4, \tag{5}$$
$$\omega = x_1 + \alpha x_2 + \beta x_3 + \alpha\beta x_4,$$

where

$$\alpha = -\frac{b_u}{4b}, \qquad \beta = \frac{b_v}{2b}, \qquad \lambda = \frac{\sqrt{-\frac{1}{6}h}}{b}, \qquad \mu = -\frac{3h}{10b^3},$$

h being the fundamental invariant $b^2(f + b_v) - \frac{1}{4}bb_{uu} + \frac{5}{16}b_u^2$ of the surface S.†

Consequently the coördinates of the *unode* when referred to the tetrahedron $P_y\, P_z\, P_\rho\, P_\sigma$ are

$$x_1 : x_2 : x_3 : x_4 = \alpha\beta : -\beta : -\alpha : 1 = -b_u b_v : -4bb_v : 2bb_u : 8b^2.$$

* M_3, pp. 297, 303.

† M_3, p. 299.

The equation of the *singular tangent plane** $\omega = 0$ becomes

$$8b^2 x_1 - 2bb_u x_2 + 4bb_v x_3 - b_u b_v x_4 = 0,$$

when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$. The polar plane of the *unode* in the complex

$$(\omega_{14} + \omega_{23}) + \alpha_{34}\omega_{34} + \alpha_{42}\omega_{42} = 0$$

is

$$8b^2 x_1 + b(2b_u - 8b\alpha_{42})x_2 + 4b(b_v + 2b\alpha_{34})x_3 + (b_u b_v - 2bb_u \alpha_{34} - 4bb_v \alpha_{34})x_4 = 0.$$

If we identify these equations, we find

$$\alpha_{34} = 0, \qquad \alpha_{42} = \frac{b_u}{2b}.$$

Hence the equation of C'' is

$$2b(\omega_{14} + \omega_{23}) + b_u \omega_{42} = 0. \tag{6}$$

The invariant of C' is $I' = -b^2$, and that of C'' is $I'' = 4b^2$. From equations (4), we find the equation of the complex $C''(u, v + dv)$ to be

$$(\omega_{14} + \omega_{23}) + \left(\alpha_{42} + \frac{\partial \alpha_{42}}{\partial v}\right)\omega_{42} = 0,$$

when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$. This will be identical with $C''(u, v)$ if, and only if, the function b_u/b is independent of v, i. e.,

$$\Omega' \equiv \frac{\partial^2 \log b}{\partial u \partial v} = 0.$$

Now this is precisely the condition that the asymptotic curves Γ' belong to linear complexes.† Hence we have the theorem:

THEOREM. *The complexes C'' which belong to the different points of the same generator of S will all coincide if, and only if, the curved asymptotic lines of the scroll belong to linear complexes.*

THE SCROLL DIRECTRIX CURVES OF THE FIRST AND SECOND KINDS

Let $\sum a'_{ij}\omega_{ij} = 0$ and $\sum a''_{ij}\omega_{ij} = 0$ be the equations of two linear complexes. Let A' and A'' be their respective invariants, and let (A', A'') be their mutual invariant. Then the equations of the directrices of the congruence common to these two complexes may be obtained by writing down the

* M_3, p. 303.

† See a paper by the author, these Transactions, vol. 15 (1914), p. 190.

conditions that must be satisfied by the coördinates of a point in order that the same plane may correspond to it in both complexes. We thus find

$$\begin{aligned}
&* + (a'_{12} - \omega_k a''_{12})x_2 + (a'_{13} - \omega_k a''_{13})x_3 + (a'_{14} - \omega_k a''_{14})x_4 = 0,\\
&(-a'_{12} + \omega_k a''_{12})x_1 + * + (a'_{23} - \omega_k a''_{23})x_3 + (a'_{24} - \omega_k a''_{24})x_4 = 0,\\
&(-a'_{13} + \omega_k a''_{13})x_1 + (-a'_{23} + \omega_k a''_{23})x_2 + * + (a'_{34} - \omega_k a''_{34})x_4 = 0,\\
&(-a'_{14} + \omega_k a''_{14})x_1 + (-a'_{24} + \omega_k a''_{24})x_4 + (-a'_{34} + \omega_k a''_{34})x_3 + * = 0,
\end{aligned}$$

where $a'_{24} = -a'_{42}$, $a''_{42} = -a''_{42}$, and where ω_1, ω_2 are the two roots of the quadratic equation

$$A'\omega^2 - (A', A'')\omega + A'' = 0.$$

For the complexes C' and C'' we find

$$\omega_1 = 2, \qquad \omega_2 = -2;$$

so that the equations of the directrices are

$$\begin{aligned}
&d : x_4 = 0, \qquad 4bx_1 - b_u x_2 + 2b_v x_3 = 0,\\
&d' : 2bx_2 + b_v x_4 = 0, \qquad 4bx_3 - b_u x_4 = 0.
\end{aligned} \tag{7}$$

The directrix of the first kind d lies in the tangent plane to the surface S at the point P_y; while the directrix of the second kind d' passes through the point P_y. The directrix of the second kind is the line joining the point P_y to the point P_τ, where

$$\tau = -2b_v z + b_u \rho + 4b\sigma.$$

As the point P_y moves over the surface S, the directrix d' generates the *directrix congruence of the second kind* G_2, one line of which passes through each point of S. The two one-parameter families of curves on S which correspond to the two ways of assembling the lines of G_2 into developables, we shall call *the directrix curves of the second kind* (C_1, C_2).

We shall now determine the directrix curves (C_1, C_2) on S. Let P_y and P_τ be displaced to $P_{y+sdu+\rho dv}$ and $P_{\tau+\tau_u du+\tau_v dv}$ respectively; the directrix d' will be displaced from the position determined by the former pair of points to that determined by the latter pair. By direct calculation, we find

$$\begin{aligned}
\tau_u &= P\ y + Q\ z + R\ \rho + S\ \sigma,\\
\tau_v &= P'y + Q'z + R'\rho + S'\sigma,
\end{aligned}$$

where

$$P = 8b^2 g + 2b_v f - 4bf_v, \quad Q = -2b_{uv}, \quad R = b_{uu} - 4b(f + b_v), \quad S = 5b_v,$$
$$P' = -b_u g, \quad Q' = -2(2bg + b_{vv}), \quad R' = b_{uv}, \quad S' = 2b_v.$$

An arbitrary point on the displaced directrix will be given by

$$\lambda(y + z\delta u + \rho\delta v) + \mu(\tau + \tau_u\,\delta u + \tau_v\,\delta v).$$

The coördinates of such a point are therefore

$$x_1 = \lambda + \mu(P\delta u + P'\,\delta v),$$
$$x_2 = \lambda\delta u + \mu(-2b_v + Q\delta u + Q'\,\delta v),$$
$$x_3 = \lambda\delta v + \mu(b_u + R\delta u + R'\,\delta v),$$
$$x_4 = \mu(4b + S\delta u + S'\,\delta v).$$

In order that this point may be on the directrix d' determined by the point P_y, its coördinates must satisfy equations (7); i. e.,

$$2b\delta u\lambda + [2b(Q\delta u + Q'\,\delta v) + b_v(S\delta u + S'\,\delta v)]\,\mu = 0,$$
$$4b\delta v\lambda + [4b(R\delta u + R'\,\delta v) - b_u(S\delta u + S'\,\delta v)]\,\mu = 0.$$

Whence

$$(8)\qquad \begin{vmatrix} \delta u & 2b(Q\delta u + Q'\,\delta v) + b_v(S\delta u + S'\,\delta v) \\ 2\delta v & 4b(R\delta u + R'\,\delta v) - b_u(S\delta u + S'\,\delta v) \end{vmatrix} = 0.$$

By means of the integrability conditions, we find

$$4bR - b_uS = -16h,$$
$$(4bR' - buS') - (4bQ + 2b_vS) = 3/2\,bC',$$
$$4bQ' + 2b_vS' = \frac{\theta'}{16}.$$

Equation (8) therefore reduces to

$$(8')\qquad 2^8\,h\delta u^2 - 3\cdot 2^7\,bC'\,\delta u\delta v + \theta'\,\delta v^2 = 0.$$

This is the differential equation of the directrix curves (C_1, C_2) on S. Its coefficients are fundamental invariants of the surface.*

Let us consider *the congruence of the first kind* (G_1) generated by the directrix d of the first kind as the point P_y moves over S. The directrix d joins the points

$$r = b_u\,y + 4bz, \qquad s = -b_v\,y + 2b\rho.$$

Let P_y be displaced to $P_{y+z\delta u+\rho dv}$. The directrix d will be displaced to the position determined by the points

$$r + r_u\,\delta u + r_v\,\delta v, \qquad s + s_u\,\delta u + s_v\,\delta v.$$

By direct calculation, we find

* M_3, p. 299.

$$r + r_u\,\delta u + r_v\,\delta v = P_1\,y + Q_1\,z + R_1\,\rho + S_1\,\sigma,$$

$$s + s_u\,\delta u + s_v\,\delta v = P_1'\,y + Q_1'\,z + R_1'\,\rho + S_1'\,\sigma,$$

where

$$P_1 = b_u + (b_{uu} - 4bf)\,\delta u + b_{uv}\,\delta v,$$

$$Q_1 = 4b + 5b_u\,\delta u + 4b_v\,\delta v,$$

$$R_1 = -8b^2\,\delta u + bu\delta v, \qquad S_1 = 4b\delta v,$$

$$P_1' = -(b_v + b_{uv}\,\delta u + \overline{b_{vv} + 2bg}\delta v), \qquad Q_1' = -b_v\delta u,$$

$$R_1' = 2b + 2b_u\,\delta u + b_v\,\delta v, \qquad S_1' = 2b\delta u.$$

The coördinates of an arbitrary point

$$\lambda'(r + r_u\,\delta u + r_v\,\delta v) + \mu'(s + s_u\,\delta u + s_v\,\delta v)$$

on this line are

$$x_1 = \lambda' P_1 + \mu' P_1',$$

$$x_2 = \lambda' Q_1 + \mu' Q_1',$$

$$x_3 = \lambda' R_1 + \mu' R_1',$$

$$x_4 = \lambda' S_1 + \mu' S_1'.$$

This point will be on the directrix d if its coördinates satisfy the equations (7), i. e., if

$$2\lambda'\,\delta v + \mu'\,\delta u = 0,$$

$$\lambda'[4(b + b_{uu} - 4b^2 f - 4b^2 b_v - 5/4b_u^2)\,\delta u + 2(2bb_{uv} - b_u b_v)\,\delta v]$$

(9)

$$+ \mu'[(5b_u b_v - 4bb_{uv})\,\delta u + 2(b_v^2 - 2bb_{vv} - 4b^2 g)\,\delta v] = 0.$$

Hence, after some simplifications and reductions, we find the equation of the directrix curves (C_1', C_2') to be

$$2^8\,h\delta u^2 - 3\cdot 2^7\,bC'\,\delta u\delta v + \theta'\,\delta v^2 = 0, \tag{8'}$$

i. e., the same as equation (8). Hence *the directrix curves of the second kind coincide with those of the first kind.*

The first equation of (9) shows that

$$-\frac{1}{2}\frac{\delta u}{\delta v} = \frac{\lambda'}{\mu'};$$

so that $\lambda' : \mu'$ will be the two roots of the quadratic equation

$$2^{10}\,h\lambda'^2 + 3\cdot 2^8\,bC'\,\lambda'\,\mu' + \theta'\,\mu'^2 = 0.$$

Thus every directrix of the first kind belongs to two developables of the

congruence formed by their totality. The points P_i $(i = 1, 2)$ where the directrix d intersects the cuspidal edges of these two developables are given by

$$\lambda_i' r + \mu_i' s = \lambda_i' (b_u y + 4bz) + \mu_i' (- b_v y + 2b\rho),$$

where $\lambda_i' : \mu_i'$ are the two roots of the above quadratic. The tangent t_k to one of the directrix curves through P_y joins this point to the point $P_{y+zdu_k+\rho dv_k}$, where $du_k : dv_k$ is a root of equation (8). The point of intersection of t_k and d is

$$ly + m\delta u_k z + m\delta v_k \rho,$$

where

$$\frac{l}{m} = \frac{b_u \delta u_k - 2b_v \delta v_k}{4b}.$$

Thus the point (t_k, d) is given by

$$(b_u \delta u_k - 2b_v \delta v_k) y + 4(z\delta u_k + \rho \delta v_k) = r\delta u_k + 2s\delta v_k = 2(- r\lambda_k' + s' \mu_k'),$$

which is the harmonic conjugate of the intersection of d with the cuspidal edge of the corresponding developable.

The cuspidal edges of the two developables to which d' belongs intersect this line in the points

$$P_i = \lambda_i y + \mu_i \tau,$$

where $\lambda_i : \mu_i$ $(i = 1, 2)$ are the roots of the equation

$$(10) \qquad \lambda^2 - \left[\frac{C'}{8} - \frac{b_u b_v}{b}\right]\lambda\mu + \left[\frac{\theta' h}{2^4 b^2} - \frac{C'^2}{2^5} - \frac{b_u b_v}{2^2 b^2}(2bb_{uv} - 3b_u b_v)\right]\mu^2 = 0.$$

Referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, the coördinates of P_1 and P_2 are

$$x_1 = \lambda_i, \qquad x_2 = - 2b_v \mu_i, \qquad x_3 = b_u \mu_i, \qquad x_4 = 4b\mu_i.$$

The canonical coördinates of these points will be

$$X = 0, \qquad Y = 0, \qquad Z = - \lambda\mu \cdot 4b\mu_i,$$

$$\omega = \lambda_i + \frac{b_u b_v}{2b} \mu_i;$$

or, in non-homogeneous form,

$$x = 0, \qquad y = 0, \qquad z = \frac{- \lambda\mu \cdot 4b}{\frac{\lambda_i}{\mu_i} + \frac{b_u b_v}{2b}} = - \frac{4b\lambda\mu}{K} = \frac{6h}{5b^3 K}\sqrt{\frac{- h}{5}},$$

where

$$K = \frac{b_u b_v}{2b} + \frac{\lambda_i}{\mu_i}.$$

Hence

$$\frac{\lambda_i}{\mu_i} = K - \frac{b_u b_v}{2b},$$

where the two values of K are the two roots of the quadratic equation

$$K^2 - \frac{C'}{8}K + \frac{2h\theta' - b^2 C'}{32b^2} = 0.* \tag{11}$$

The coefficients of this equation are invariants, as was to be expected. We may state these results in the form of the following theorem:

THEOREM. *As a point P_y describes a directrix curve (C_1 or C_2) on a scroll, its directrices (d and d') of both kinds describe developables. Let the tangent to one of the directrix curves be constructed at the point, as well as the asymptotic tangents at this point. The harmonic conjugate of the first line with respect to the other two intersects the directrix of the first kind in a point on the cuspidal edge of the developable which it describes. The non-homogeneous canonical coördinates of the points in which the directrix of the second kind intersects the edges of regression of the respective developables to which it belongs are*

$$x = 0, \qquad y = 0, \qquad z = \frac{6h}{5b^3 K}\sqrt{\frac{-h}{5}},$$

where K is a root of the invariant equation (11).

The asymptotic tangent to the surface S at the point $(\alpha y + \beta\rho)$ of the generator Γ'' joins this point to the point $(\alpha z + \beta\sigma)$.† Hence the coördinates of an arbitrary point on the *osculating hyperboloid* H are

$$x_1 = \lambda\alpha, \qquad x_2 = \mu\alpha, \qquad x_3 = \lambda\beta, \qquad x_4 = \mu\beta.$$

Therefore the equation of H referred to $(P_y P_z P_\rho P_\sigma)$ is

$$x_1 x_4 - x_2 x_3 = 0. \tag{12}$$

The directrix d' intersects H in the points $(-b_u b_v, -4bb_v, 2bb_u, 8b^2)$ and P_y. But the former is the *pinch point* of the Cayley cubic scroll osculating S at the point P_y (see equation (5)).

If $Y = y + l_\rho$, where l is an arbitrary constant, the canonical equations of S considered as the locus of P_Y are‡

$$\eta_{uu} + 2B\eta_v + F\eta = 0, \qquad \eta_{vv} + G\eta = 0, \tag{13}$$

where

$$B = b + \frac{ll_v - \frac{1}{2}l^2 f_v}{1 + l^2 g},$$

$$F = \frac{f + lf_v + 2l^2 gb_v + l^2 bg_v}{1 + l^2 g} + \frac{l^2 g_v(b_v - \frac{1}{2}lf_v)}{(1 + l^2 g)^2}, \tag{14}$$

$$G = g + \frac{lg_v + \frac{1}{2}l^2 g_{vv}}{1 + l^2 g} + \frac{l^4 g_v}{(1 + l^2 g)^2}.$$

* Cf. M_2, pp. 117, 118, 119.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 146.

‡ M_3, pp. 311, 312.

The fundamental semi-covariants of this system of equations are

$$\eta = \frac{1}{\lambda}(y + l\rho), \qquad Z = \frac{1}{\lambda}(z + l\sigma),$$

$$\Sigma = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) z + \left(1 - \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) \sigma \right],$$

$$\Sigma = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) z + \left(1 - \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) \sigma \right].$$

The equations of the directrix d' of the point P_Y are, according to equations (7),

$$\begin{aligned} 2BX_2 + B_v X_4 &= 0, \\ 4BX_3 - B_u X_4 &= 0, \end{aligned} \tag{15}$$

the tetrahedron of reference being that determined by (η, Z, P, Σ). From the definition of the coördinates of the point $Q(x_1 y + x_2 z + x_3 \rho + x_4 \sigma)$ and the above expressions for (η, Z, P, Σ), we find the following equations of transformation between the coördinate systems determined by the semi-covariants (η, Z, P, Σ) and (y, z, ρ, σ) respectively:

$$\begin{aligned} \omega X_1 &= Qx_3 - Q_1 x_1, & \omega X_2 &= Qx_4 - Q_1 x_2, \\ \omega X_3 &= lx_1 - x_3, & \omega X_4 &= lx_2 - x_4, \end{aligned}$$

where

$$Q = -\left(lg + \tfrac{1}{2}\frac{l^2 g_v}{1 + l^2 g} \right), \qquad Q_1 = \left(1 - \tfrac{1}{2}\frac{l^2 g_v}{1 + l^2 g} \right),$$

and where ω is a factor of proportionality.

If we apply this transformation to equations (15), we shall find the equations of the directrix d' of the point P_Y to be

$$l(b_v x_2 + \beta_2 x_4) + (2bx_2 + b_v x_4) = 0,$$

$$l\left(4x_1 - \frac{\partial \log B}{\partial u} x_2 \right) - \left(4x_3 - \frac{\partial \log B}{\partial u} x_4 \right) = 0,$$

where

$$\beta_2 = 2bg - f_v,$$

when referred to the tetrahedron $(P_y P_z P_\rho P_\sigma)$. Now the equation of the surface generated by d' as P_y moves along Γ'' is found by eliminating the parameter l between these equations. This surface turns out to be a quartic which we shall denote, for brevity, by Q.

Equations (5) and (7) show that the locus of the *unodes* of the Cayley cubic scrolls osculating S along Γ'' is situated on the surface Q. This locus is the twisted cubic whose parametric equations are*

* M_3, p. 312.

$$x_1 = -[b_u + b_{uv}\,l + \tfrac{1}{2}(b_{uvv} + 2gb_u)l^2][b_v + (b_{vv} + 2bg)\,l],$$

$$x_2 = -[4b + 4b_v\,l + 2(b_{vv} + 2bg)l^2][b_v + (b_{vv} + 2bg)\,l],$$

$$x_3 = [b_u + b_{uv}\,l + \tfrac{1}{2}(b_{uvv} + 2gb_u)l^2][2b + b_v\,l],$$

$$x_4 = [4b + 4b_v\,l + 2(b_{vv} + 2bg)l^2][2b + b_v\,l].$$

This twisted cubic will be an asymptotic curve on Q, if its osculating plane at the point

$$\pi = -b_u\,b_v\,y - 4bb_v\,z + 2bb_u\,\rho + 8b^2\sigma \quad \text{(see (5), (7))}$$

contains the corresponding generator d' of Q. Now if we determine the plane osculating this curve at π, we shall find, after a rather long calculation, that it can pass through d' only when the invariant

$$\Delta \equiv \frac{3b\theta'}{2^4}(2b_u\,\theta' - b\theta'_u)$$

vanishes. Hence we have the theorem:

THEOREM. *The lines of the congruence (G_2) can be assembled into a one-parameter family of quartic scrolls having the generators of S as directrices. The twisted cubic which is the locus of the unodes of the Cayley cubic scrolls osculating S along Γ'' will be an asymptotic curve on Q only when the invariant Δ vanishes.*

LINES COMMON TO THE COMPLEXES $C'(u, v)$, $C''(u, v)$, $C'(u, v + dv)$, $C''(u + du, v)$

We shall first determine the complexes $C'(u, v + dv)$, $C''(u + du, v)$ which belong to the points $P_{y+\rho dv}$ and P_{y+zdu} respectively. Let P_y receive displacements to P_{y+zdu} and $P_{y+\rho dv}$, and let $(\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma})$ and $(\bar{\bar{y}}, \bar{\bar{z}}, \bar{\bar{\rho}}, \bar{\bar{\sigma}})$ be the semi-covariants corresponding to these new positions. The values of these semi-covariants in terms of those of the position P_y are given by the following equations:

$$(17)\qquad \begin{aligned} \bar{y} &= y + zdu + \cdots, & \bar{z} &= z - (fy + 2b\rho)\,du + \cdots,\\ \bar{\rho} &= \rho + \sigma du + \cdots, & \bar{\sigma} &= \sigma + (\beta_2\,y + \beta_4\,\rho)\,du + \cdots,\\ \bar{\bar{y}} &= y + \rho dv + \cdots, & \bar{\bar{z}} &= z + \sigma dv + \cdots,\\ \bar{\bar{\rho}} &= \rho - gydv + \cdots, & \bar{\bar{\sigma}} &= \sigma - gzdv + \cdots \end{aligned}$$

where

$$\beta_2 = 2bg - f_v, \qquad \beta_4 = -(f + 2b_v).$$

From these, by the method used in determining equations (4), we find the

following equations of transformation between the coördinate systems $(\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma})$, $(\bar{\bar{y}}, \bar{\bar{\rho}}, \bar{\bar{z}}, \bar{\bar{\sigma}})$ and, (y, z, ρ, σ):

$$
\begin{aligned}
x_1 &= \bar{x}_1 + (-f\bar{x}_2 + \beta_2 \bar{x}_4)\,du = \bar{\bar{x}}_1 - g\bar{\bar{x}}_3\,dv, \\
x_2 &= \bar{x}_2 + \bar{x}_1\,du \qquad\qquad\quad = \bar{\bar{x}}_2 - g\bar{\bar{x}}_4\,dv, \\
x_3 &= \bar{x}_3 + (-2b\bar{x}_2 + \beta_4 \bar{x}_4)\,du = \bar{\bar{x}}_3 + \bar{\bar{x}}_1\,dv, \\
x_4 &= \bar{x}_4 + \bar{x}_3\,du \qquad\qquad\quad = \bar{\bar{x}}_4 + \bar{\bar{x}}_2\,dv.
\end{aligned} \tag{18}
$$

whence

$$
\begin{aligned}
\bar{\bar{x}}_1 &= x_1 + gx_3\,dv, & \bar{\bar{x}}_3 &= x_2 + gx_4\,dv, \\
\bar{\bar{x}}_3 &= x_3 - x_1\,dv, & \bar{\bar{x}}_4 &= x_4 - x_2\,dv, \\
\bar{x}_1 &= x_1 + (fx_2 - \beta_2 x_4)\,du, & \bar{x}_2 &= x_2 - x_1\,du, \\
\bar{x}_3 &= x_3 + (2bx_2 - \beta_4 x_4)\,du, & \bar{x}_4 &= x_4 - x_3\,du,
\end{aligned} \tag{18'}
$$

and

$$
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} - (\omega_{13} + f\omega_{42})\,du, \qquad \bar{\omega}_{23} = \omega_{23} + (\beta_4\,\omega_{42} - \omega_{13})\,du, \\
&\qquad\qquad \bar{\omega}_{42} = \omega_{42} + (\omega_{14} + \omega_{23})\,du, \\
\bar{\bar{\omega}}_{14} &= \omega_{14} + (+g\omega_{34} - \omega_{12})\,dv, \qquad \bar{\bar{\omega}}_{23} = \omega_{23} + (\omega_{12} - g\omega_{34})\,dv, \\
&\qquad\qquad \bar{\bar{\omega}}_{34} = \omega_{34} + (\omega_{23} - \omega_{14})\,dv.
\end{aligned} \tag{18''}
$$

From these we find that the equations of the complexes $C'(u, v + dv)$ and $C''(u + du, v)$ become

$$(-b_v\,\omega_{34} - b\omega_{14} + b\omega_{23}) + dv\,(2b\omega_{12} - \beta_2\,\omega_{34}) = 0,$$

$$[2b\,(\omega_{14} + \omega_{23}) + b_u\,\omega_{42}] + du\,[3b_u\,(\omega_{14} + \omega_{23}) - 4b\omega_{13} - (b_{uv} + \alpha_3)\,\omega_{42}] = 0,$$

where

$$\alpha_3 = 2\,(2bf + 2bb_v - b_{uu}).$$

Then clearly the lines required are those common to the four complexes

$$
\begin{aligned}
&1.\quad C' : -b_v\,\omega_{34} - b\omega_{14} + b\omega_{23} = 0, \\
&2.\quad C'' : 2b\,(\omega_{14} + \omega_{23}) + b_u\,\omega_{42} = 0, \\
&3.\quad 2b\omega_{12} - \beta_2\,\omega_{34} = 0, \\
&4.\quad 3b_u\,(\omega_{14} + \omega_{23}) - 4b\omega_{13} - (b_{uu} + \alpha_3)\,\omega_{42} = 0.
\end{aligned} \tag{19}
$$

The lines common to these complexes must intersect d and d'. The points

$$p = b_u\,y + 4bz, \qquad q = -b_v\,y + 2b\rho$$

are on d, and the points y, $r = -2b_v\,z + b_u\,\rho + 4b\sigma$ are on d'. Hence the coördinates of an arbitrary point on d will be

$$
\begin{aligned}
x_1 &= \lambda b_u - \mu b_v, \qquad x_2 = 4b\lambda, \\
x_3 &= 2b\mu, \qquad\quad x_4 = 0;
\end{aligned}
$$

and those of an arbitrary point on d' will be

$$x_1' = \lambda', \qquad x_2' = -2b_v\mu',$$
$$x_3' = b_u\mu', \qquad x_4 = 4b\mu'.$$

Therefore the coördinates of the line joining these points will be

$$\begin{aligned}
\omega_{12} &= -4b\lambda\lambda' - 2b_u b_v \lambda\mu' + 2b_v^2 \mu\mu',\\
\omega_{13} &= -2b\mu\lambda' + b_u^2 \lambda\mu' - b_u b_v \mu\mu',\\
\omega_{14} &= 4bb_u \lambda\mu' - b_u b_v \mu\mu',\\
\omega_{23} &= 4bb_u \lambda\mu' + b_u b_v \mu\mu',\\
\omega_{34} &= 8b^2 \mu\mu',\\
\omega_{42} &= -16b^2 \lambda\mu'.
\end{aligned}$$

In order that this line may belong to the second pair of complexes (19), the equations

$$\begin{aligned}
(20) \qquad & -2^7 b\lambda\lambda' - 2^6 b_u b_v \lambda\mu' + \theta'\mu\mu' = 0,\\
& ab\mu\lambda' + 2^4 h\lambda\mu' + b_u b_v \mu\mu' = 0.
\end{aligned}$$

must be satisfied. Thus we find

$$\lambda^2 : \mu^2 = -\frac{\theta'}{64} : h,$$

$$\lambda' : \mu' = -\left(b_u b_v + 2^4 h\frac{\lambda}{\mu}\right) : 2b.$$

Clearly, when θ' and h are both equal to zero, the four complexes have a pencil of lines in common; this pencil is determined by d and the *pinch-point* of the Cayley cubic scroll osculating S at P_y. If θ' and h are different from zero, then the four complexes have two lines in common which intersect d in two points which form a harmonic group with the points p and q; and their intersections with d' are harmonic conjugates with respect to H (i. e., with respect to the pinch-point of the Cayley cubic scroll osculating S at P_y, and P_y). These results may be combined in the following theorem:*

THEOREM. *Consider a scroll whose flecnode curve consists of two distinct branches, and a point on S for which the invariant h is different from zero. The four complexes* $C'(u, v)$, $C''(u, v)$, $C'(u, v + dv)$, $C''(u + du, v)$ *have two lines in common. Let there be marked on the directrix of the first kind its intersections p and q with the asymptotic tangents of* P_y. *The two lines common to the four complexes intersect the directrix of the first kind in two points which are harmonic conjugates with respect to p and q. These two lines intersect the directrix of the second kind in two points which are harmonic conjugates with respect to the pair of points in which this directrix intersects the osculating hyper-*

* Cf. M_2, pp. 97, 98.

boloid. If the two branches of the flecnode curve on S coincide and the invariant h is different from zero, the four complexes have one line in common; if, however, the invariant h is equal to zero, the four complexes have a pencil of lines in common. This pencil is determined by the pinch-point of the Cayley cubic scroll osculating S at the point and the directrix of the first kind.

SURFACES FOR WHICH Ω' VANISHES AND θ' IS DIFFERENT FROM ZERO

Several of the preceding results assume a simpler form when the invariant $\Omega' \equiv (\partial^2 \log b)/(\partial u \partial v)$ vanishes. Because, if Ω' vanishes, it is possible to find a transformation of the form

$$y = \frac{1}{\sqrt{\frac{d\bar{u}}{du} \cdot \frac{d\bar{v}}{dv}}}, \qquad \bar{u} = \int \sqrt[3]{U}\, du, \qquad \bar{v} = \int \frac{dv}{\sqrt[3]{V}},$$

where U and V are functions of u alone and v alone respectively, that transforms equations (1) into the following system:*

$$y_{uu} + 2y_v + fy = 0, \qquad y_{vv} + gy = 0,$$

where f is a function of u alone and g is a constant. Thus the invariant h is a function of u alone and the invariant θ' is a constant. The equation (8) becomes

$$\overline{dv}^2 - \phi(u)\,\overline{du}^2 = 0,$$

or

$$\bar{U} \pm \bar{V} = \text{const.},$$

where

$$\bar{U} = \int \sqrt{\phi(u)}\, du, \qquad \bar{V} = v.$$

Hence the directrix curves (C_1) and (C_2) form a conjugate system on S. From equations (7), (14), (21), it is easily found that the equations of d and d' of the point P_Y become

$$d : x_4 - lx_2 = 0, \qquad x_1 + lgx_3 = 0,$$
$$d' : x_2 + lgx_4 = 0, \qquad lx_1 - x_3 = 0,$$

when referred to the coördinate system $(P_y P_z P_\rho P_\sigma)$ of the point P_y. These equations show that d and d' belong to the same regulus of a quadric surface (H_1). The equation of (H_1) is

$$x_1 x_2 + g x_3 x_4 = 0.$$

Now (H) and (H_1) intersect in the two generators Γ''' and Γ_1''' (Γ_1''' being the generator on S_1, the derived ruled surface of S, which corresponds to Γ''' on S*) which belong to the same regulus; they must, therefore, intersect in

* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 124.

two other generators which belong to the complementary regulus. It is easy to see that these generators pass through the points $y \pm 1/(\sqrt{-g})\rho$ on Γ''. The flecnodes on Γ'' are given by the factors of the quadratic covariant*

$$u'_{12}\rho^2 - u'_{21}y^2 + (u'_{11} - u'_{22})y\rho,$$

where

$$u'_{11} = -4f, \quad u'_{12} = -8b, \quad u'_{22} = -4(f + 2b_v),$$
$$u'_{21} = -4(f_v - 2bg) = 4(b_{vv} + 2bg).$$

Since the asymptotic curves Γ'' belong to linear complexes, the surface S has two straight-line directrices.† Hence the second pair of generators of intersection of (H) and (H_1) must coincide with these directrices (δ, δ') of S.

If the invariants h, θ', C' all vanish, the directrix curves (C_1) and (C_2) become indeterminate. But the vanishing of these invariants is completely characteristic of a Cayley cubic scroll.‡ In fact, from the second of equations (21), we find

$$y = U_1 v + U_2$$

where U_1 and U_2 are functions of u alone. If this value of y be substituted in the first of equations (21) and the resulting equations

$$\frac{d^2U_1}{du^2} = 0, \qquad \frac{d^2U_2}{du^2} + 2U_1 = 0,$$

be integrated, we find

$$y = (au + b)v + \left(-\frac{a}{3}u^3 - bu^2 + cu + d\right),$$

where a, b, c, d are constants. Hence the equation of S is

$$3(y_2y_3 - y_1y_4)y_4 = 2y_3^3.$$

In this case we readily find the following equations of transformation between the coördinate systems (y, z, ρ, σ) and the fixed system of reference (y_1, y_2, y_3, y_4):

$$\begin{aligned} y_1 &= (-\tfrac{1}{3}u^3 + uv)x_1 + (-u^2 + v)x_2 + ux_3 + x_4, \\ y_2 &= (-u^2 + v)x_1 - 2ux_2 + x_3, \\ y_3 &= ux_1 + x_2, \qquad y_4 = x_1, \end{aligned} \tag{22}$$

whence

* M_3, p. 313.

† See a paper by the author, these Transactions, vol. 15 (1914), p. 171. Since, apart from Γ'' and Γ''', H and H_1 intersect along two generators for which $(x_3/x_4)^2 = -1/g$, and (δ, δ') are the only generators on H for which this is true, H and H_1 have (δ, δ') in common.

‡ M_3, p. 307.

$$x_1 = y_4, \qquad x_2 = y_3 - uy_4,$$
$$x_3 = y_2 + 2uy_3 - (u^2 + v)\,y_4,$$
$$x_4 = y_1 - uy_2 - (u^2 + v)\,y_3 + \left(\frac{u^3}{3} + uv\right) y_4.$$

The complexes C' and C'' now become (when referred to the fixed system of coördinates)

$$\omega_{14} - \omega_{23} - 2v\omega_{34} = 0,$$
$$\omega_{14} + \omega_{23} - 2u\,(u\omega_{34} - \omega_{42}) = 0.$$

The equations of the directrices are

$$d : y_4 = 0, \qquad y_1 - uy_2 - (u^2 + v)\,y_3 + \left(\frac{u^3}{3} + uv\right) y_4 = 0,$$
$$d' : y_3 - uy_4 = 0, \qquad y_2 + 2uy_3 - (u^2 + v)\,y_4 = 0.$$

All of the directrices d are situated in the singular tangent plane $y_4 = 0$; and all of the directrices d' pass through the *unode* $(1, 0, 0, 0)$. These results may be stated in the form of the theorem:

THEOREM. *If a scroll S have distinct straight line directrices, the rays of the directrix congruences of both kinds which correspond to the points of a generator Γ'' of S belong to the same regulus of a quadric (H_1); and this quadric contains the directrices (δ, δ') of the surface S itself. The directrix curves (C_1) and (C_2) become indeterminate when, and only when, the surface S is a Cayley cubic scroll. In this case the directrix congruences of both kinds degenerate; that of the first kind consists of the net of lines in the singular tangent plane, while that of the second kind consists of the sheaf of lines through the unode.*

McGILL UNIVERSITY, MONTREAL, CANADA.